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ABSTRACT

This material is the result of 1 year of experimental work during the 1976-77 school year with pupils at the fifth-grade level. Once a week students participated in half-hour, small-group sessions in addition to their regular classroom mathematics period. It is felt that the activities described might be appropriate for additional grade levels. It is hoped that teachers who wish to direct part of their energies to improving the quality of life for weaker students will be stimulated to experiment with some of the ideas included here. The focus is on games, puzzles, and detective stories that are grouped under: (1) Hand Calculator; (2) Minicomputer; and (3) Detective Stories. (MP)

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Comprehensive School Mathematics Program
CEMREL, Inc.
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CSMP Mathematics for the Intermediate Grades

Math Play Therapy

Volume II

ED226947

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MATH PLAY THERAPY

by Frederique Papy

with the collaboration of Cille Smith

"Math Play Therapy", Volume 2, is the result of one year of experimental work during the school year 1976-77 with students who were at the fifth grade level. Once a week the students participated in small group sessions. These one-half hour sessions were in addition to their regular classroom mathematics period. Perhaps the activities described herein are also appropriate for other grade levels.

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INTRODUCTION

At the heart of the CSMP curriculum is our belief that each child at any level of ability can and should be given the chance to play an important role in group exploration of the world of mathematics. To that end, we attempt to develop a curriculum and a way of teaching that will actualize this belief. Perhaps an analogy to improvisational theater, *commedia dell'arte*, will help us clarify the ways we have implemented this aim in our collective work with the children.

For such drama, an outline is written which offers actors the possibility of improvising in many different directions depending on their particular interests and talents. This is not the kind of theater that highlights one or two stars and expects all the other players to act in such a way that the attention is directed to the stars. Rather it is theater in which many people share the spotlight as the emphasis shifts back and forth across the stage, giving much the same effect as an orchestra with its beautiful and greatly varied tonal colors. In the same manner, CSMP lesson outlines are written to make use of different languages and strands of mathematics which enrich a situation and offer important opportunities to all kinds of children whose skills and interest vary widely. Just as the director of improvisational theater rejects a leadership role to become a facilitator for others, so the person who teaches in the CSMP spirit strives always to be less teacher than enabler -- one who removes obstacles in order to include others in the action, rather than one who controls the action.

In spite of the greatly expanded participation that is made possible in our mathematical drama through the use of non-verbal languages and previously neglected areas of mathematical thinking, there are still some young actors who are unable to play an important role. They do not possess enough numerical knowledge to contribute significantly to the collective work. Nor can the encouragement and ingenuity of a facilitator unlock the door to full participation for those children in a situation where many other players are more adequately prepared or more generously gifted. And so these actors come to recognize that they are only playing walk-on roles, which appear as shadows flickering across a stage filled with vibrant personalities.

Should we really be surprised when the children from this group react by "turning off" in passive ways such as daydreaming or doodling, or in active ways such as playing the clown or the despot to disrupt a situation they cannot understand? In fact, these kinds of responses often indicate children struggling for self-preservation and self-fulfillment, misdirected and inadequate though their responses may be.

Recognizing that the impulse for self-fulfillment arises from life itself, we refused to accept walk-on roles as the outer limits of possibility for these children. So we chose to act with faith in them and in ourself. We transplanted the essence of *commedia dell'arte* to math play therapy sessions where we hoped to create an environment that would more adequately nourish growth. To free our young players from pressure, criticism, and classroom distraction, we arranged to meet in a small room and work with just a few of them at a time. For each group, we tried to select students who had similar abilities and complementary personalities. So began, in October 1975, a weekly experiment that was to affect us, the children, and our relationships to each other.

In the spirit of *commedia dell'arte*, we began designing games and dramas at a more appropriate level for these students by decreasing the numerical sophistication without reducing it to triviality. We attempted to maintain the richness and variety of approaches that were beneficial in the collective lessons, but to further alter the focus by giving the players almost complete responsibility for the improvisation within this simplified framework. We were convinced that only by experiencing greater freedom and responsibility of true leadership could they grow and come closer to attaining their potential. So we gave ourself up to the enjoyment of the unfolding drama, freed to recognize and respond to these young persons as they revealed themselves in their play.

When we invited the students to join us in the first math play therapy session, we felt some hesitancy and mistrust. This was openly expressed by one student when he asked, "Why do we have to come? Is it because we are dumb?" It is very significant to us that such a question has never been asked again. The students understood that what they were offered in the little room was different in spirit from the usual remedial work and did not infer stupidity in any way. As the year progressed, it became the practice of the home room teacher to allow the remaining participants of a group to select a substitute player when a regular member was absent. Their classmates considered the selection an honor to be coveted and frequently begged to be allowed to take part. Perhaps an observation of one of the fourth grade teachers most clearly illustrates the dramatic change in attitude. At the beginning of the year, she reported that her slower students only enjoyed two intervals in the school day and those were recess and lunchtime. Later they added another favorite to the list -- mathtime! Already the freedom to respond more naturally in the small groups was improving their general receptivity to mathematics.

Although it was not an instantaneous transformation, the behavior and attitudes of some children changed rather quickly. But for the children whose egos were severely damaged, the process was a slower one. In addition, there was still so much ground for them to cover that even in the math play therapy sessions they frequently played only a minor role in the improvisation. Consequently, they were often tense and defensive, even disruptive. Our response was to accept such behavior as natural at this stage of development and to allow them to overcome it themselves as long as it was possible without being unfair to the entire group. If it was necessary to ask a student to leave a session, we did so calmly with the reminder that the student would be included the following week. Slowly and without outside pressure, the children found the inner resolve to change their behavior in order to meet challenging situations in more positive ways.

From the first day we met the students in the small room, we implemented our decision to give them the leadership for the group. We were convinced that offering direct assistance would deprive them of both the opportunity and the joy of making their own discoveries and would reinforce their belief that they could not solve a problem or complete a task unaided. We intuitively felt that a non-directive approach with our young friends might offer them a better hope of cultivating true self-esteem by allowing them to experience their own potential more completely.

Week after week we saw increasing evidence that the students' responses were changing. Their fear of failure, which at first inhibited new ideas and cast a gloomy spell on several sessions, was soon replaced by an eagerness to experiment that brought enjoyment and relaxation to the group. Their ideas for approaching a problem became more and more creative. When we compared sessions in which the same basic outline was used for two different groups, we recognized that the more creative and innovative of the two was the one in which the group had been given less direct assistance. We frequently discovered that the best solutions were those we had not anticipated. The fact was that almost every time we interferred in the group process, we impoverished and narrowed the possibilities.

Faced with such a potent reminder, we gradually became more aware. The more we payed attention, the more we learned, and not only about the children themselves. Because they were developing at a slower pace, these young people were in a unique position to reveal the learning process to us.

How very well the *commedia dell'arte* fit the needs of these young people who were so hemmed in by the ordinary procedures in a classroom. Their resistance to conformity became an asset in improvisation. They were freed from the rigidity of paper and pencil, where their responses had so frequently been judged as inadequate Body language, a natural and eloquent means of expression for them, was not only accepted but encouraged. When they were able to make real breakthrough discoveries, it was with a release of energy that was explosive in its intensity. The memory of one particular breakthrough still has the power to lift our spirits. It occurred near the end of a session when one student intuitively felt he could locate a winning move in a game the group was playing. He concentrated very intensely for some moments before his face lit up with the anticipation of success. Springing from his seat, he rushed to the Minicomputer boards, made the winning move, and spun across the room in a flashing pirouette of joy and satisfaction. The entire group responded with enjoyment and expansiveness.

As the players became more relaxed, free, and cheerful, their innate powers of concentration improved and their numerical intuition began to bloom.

We discovered that several of these students had good intuition in many different areas of mathematics. Even in the collective lessons we began to see them flower. Members of the small groups contributed ideas to the regular class work with increasing frequency. Occasionally one or another of them actually played a starring role in the big group.

At the end of two years we are in the midst of a fascinating process of change. In our work with improvisation, we have only begun to scratch the surface of a rich store of possibilities. We hope to have the opportunity to continue our exploration by working with most of the same children for two more years.

This book presents a selection of games, puzzles, and detective stories recalled from our records of the year's work. Each one shows the result of a particular group's improvisation on a specific theme. Altogether they constitute only a few of the ways our students responded in the *commedia dell'arte.*

Although it is incomplete, we present it as a small ray of light on an area that has been discouragingly dark for too many years. We hope other teachers who wish to direct part of their energies to improving the quality of life for our weaker students will be stimulated to experiment with some of the ideas included here and in a similar free spirit.

1

HAND-CALCULATOR

1.1 Introduction

The focus of this chapter is on a series of very simple games played with the help of the hand-calculator[*]. We had three goals in mind when these games were designed during the fall of 1975.[○]

1) To offer a simple situation in which students can explore and attain some familiarity with the hand-calculator alone before being asked to use it in combination with other tools and languages.
2) To allow time for free play in open situations where there are no wrong moves in order to help students build confidence in themselves and in the hand-calculator.
3) To lay the groundwork for two Minicomputer games by using hand-calculator activities which are, in some sense, easier versions of those games.

Perhaps for the first time in their lives, these students had the opportunity to explore large numbers in a situation they were able to control, and their interest was captured immediately. The games served a valuable purpose in two ways. First, they were beneficial in giving the students a better mental picture of the segment of the number line from 0 to 1,000. Second, they helped increase the students' abilities to estimate the distance between two numbers. After some months, the students were ready for more complex activities involving other tools and languages. These beginning games had played their transitory role and were dropped.

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[*] It is imperative that the hand-calculator used for these games have algebraic logic; that is, respond to instructions in the order they are usually written.

[○] See Math Play Therapy Volume 1, Chapter 1.

In the middle of the second year, however, the hand-calculator games made a new appearance in two different circumstances. The collective lessons about decimals were difficult for us because the slower students lacked the basic experience which would enable them to take an active role in these lessons. So we decided to try a new version of the hand-calculator game using decimals rather than whole numbers. The result of this work is presented in Section 1.3.

At about the same time our two weakest students, Barry and Ralph*, still exhibited a very fuzzy mental image of the number line. We thought they needed more experience in playing the simpler versions of the hand-calculator games to enable them to make true progress with more advanced ideas. Section 1.2 gives a report of two of their sessions with these games and suggests that, weak as they are, they still have potential and should not be counted out. They are capable of working with mathematical problems in meaningful ways and of realizing improvement and growth.

1.2 Games With Whole Numbers

Pairing Barry and Ralph in the small groups was a natural choice for several reasons. They are at about the same level in basic computational skills and understanding of mathematical concepts. In addition, they are friends and have complementary personalities. Barry has slow, steady speech and movements and is passive to the point of being withdrawn. However, he is not unreachable and responds warmly to another person's lead. He often chooses extroverted children as his closest friends. Barry exhibits a strong ego and is able to accept his limitations without giving up on his potential. He has a very pleasant, quiet way about him and sometimes surprises one with a bit of mischievous humor. Ralph is the extrovert with a mercurial temperament that is a perfect balance for Barry's evenness. He can be sullen and dejected at the beginning of the morning and become open and animated before the lunchtime break. He has quick and exaggerated patterns of movement and speech and is nothing if not impulsive, with an imagination that is capable of great flights of fantasy. Ralph will sometimes exhibit such hostility that one is alarmingly aware of a great potential for violence, but in his sunnier moods he can be fascinating to watch. His interactions with others can be very kind and gentle. Although he appears to be more sensitive than Barry about his learning problems, he also exhibits great determination and optimism when he is placed in an encouraging environment.

---

*Fictional names.

Now let us focus on the boys' initial experience with a hand-calculator game in its simplest form.

March 11

T: Today we will play a new game.

Display 37 on your hand-calculator. You will play in turn pressing

$\boxed{+}$ $\boxed{\text{any number}}$ $\boxed{=}$ or $\boxed{-}$ $\boxed{\text{any number}}$ $\boxed{=}$.

The first player to reach 500 will be the winner.

Ralph plays first and presses $\boxed{+}$ $\boxed{9}$ $\boxed{=}$.

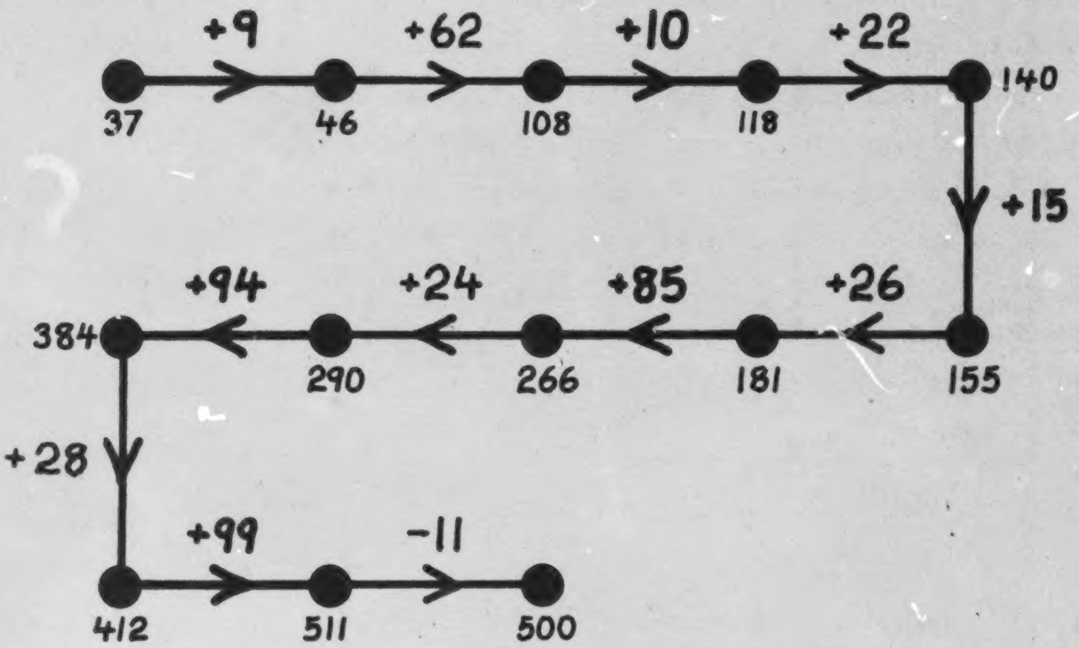
T: Barry, which number is on your display?

B: 46.

T: Now it is Barry's turn.

Barry presses $\boxed{+}$ $\boxed{62}$ $\boxed{=}$. Ralph reads the result, 108, and presses $\boxed{+}$ $\boxed{10}$ $\boxed{=}$. The game continues in this manner as indicated in the following arrow picture and ends after 12 moves. Barry wins.

Red = Ralph Blue = Barry

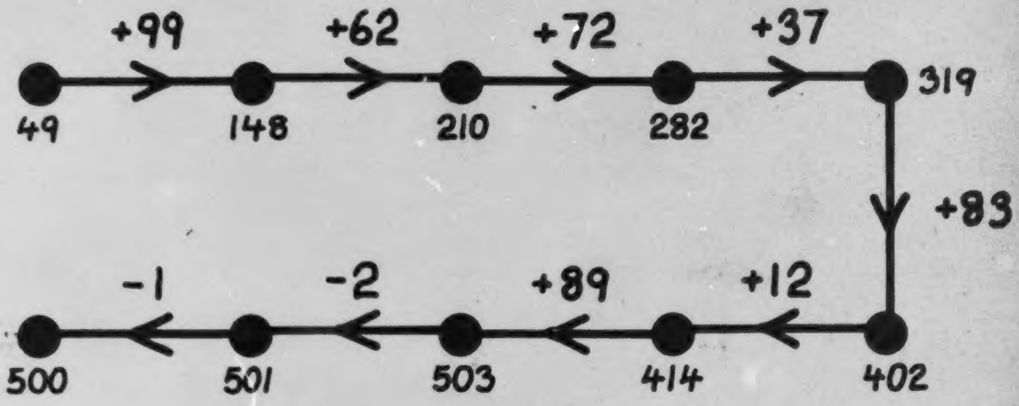

+9
+62
+10
+22
37
46
108
118
140
+15
+94
+24
+85
+26
384
290
266
181
155
+28
+99
-11
412
511
500

NOTE: In all the pictures of this section, the red arrows represent Ralph's moves and the blue arrows represent Barry's moves.

The teacher does not make any comment and does not register approval or disapproval with her face during the game. She does not take any notes. She concentrates on the activity itself and on observing the reactions of the children. Records of the game are kept by the observer.

Ralph's first three moves are very tentative. His next three moves are more resolute . . . the last one leading to Barry's victory. Barry seems to play in a more mechanical way, not taking any risks. But he knows how to go from 511 to 500 in one move. His face brightens slightly before he presses $\boxed{-}$ $\boxed{11}$ $\boxed{=}$. He anticipates his victory.

T: Let's play a second game, starting at 49. The goal will be 500 again.



The second game ends in nine moves. The two boys play more or less in the same spirit as before. At the very end, Barry misses the goal by going from 503 to 501 and leaves an easy victory for Ralph.

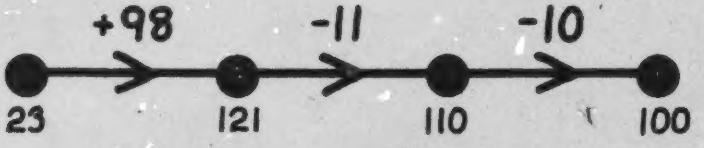
T: Let's change the rules of the game.

Ralph will play all by himself, starting at 23. As before he can press

$\boxed{+}$ $\boxed{\text{any number}}$ $\boxed{=}$ or $\boxed{-}$ $\boxed{\text{any number}}$ $\boxed{=}$.

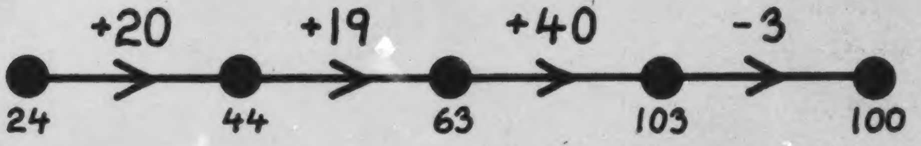
He should try to reach 100 with the smallest number of moves. Barry will follow Ralph and play a game by himself on his hand-calculator.

Ralph plays:



T: Ralph has reached the goal in three moves. His score is 3. It's your turn, Barry. Start at 24 and try to reach 100 quickly.

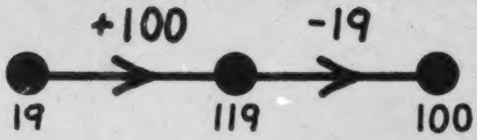
Barry plays:

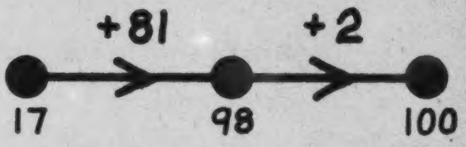


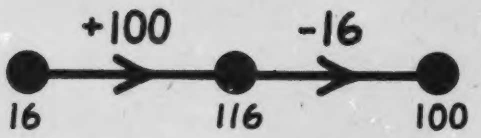
T: Your score is 4. Who is the winner?

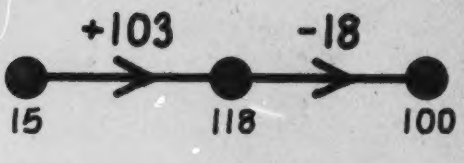
Ralph: Me.

A similar game is played three more times. The teacher suggests a goal of 200 for the last game.


+100
-19
19
119
100


+81
+2
17
98
100


+100
-16
16
116
100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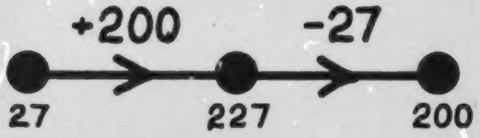
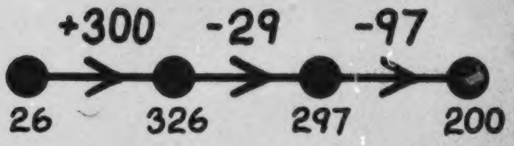
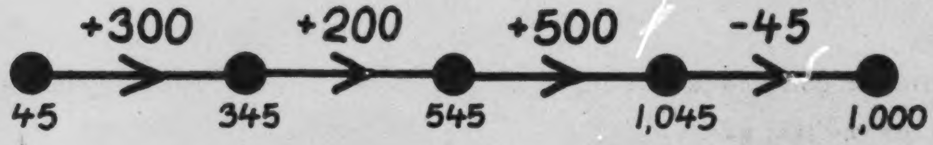
-18
15
118
100


+200
-27
27
227
200


+300
-29
-97
26
326
297
200

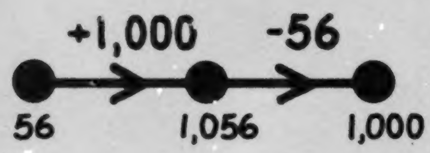
The teacher doesn't comment on the results of the games or the way the children play. Clearly, Ralph has found a strategy for solving the problem in two moves while Barry is still tentative in his plays.

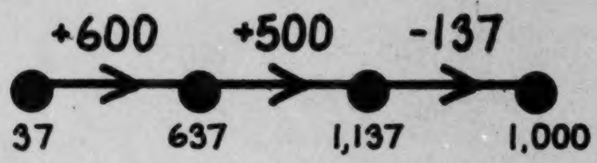
T: Let's play our first game again, starting at 45. The winner will be the player who reaches 1,000 first.


+300
+200
+500
-45
45
345
545
1,045
1,000

The game ends in four moves. A bold move by Ralph leads to a victory for Barry, who knows how to reach the goal when he is above it.

T: Now let's try our second game and see who can reach 1,000 in the smallest number of moves.

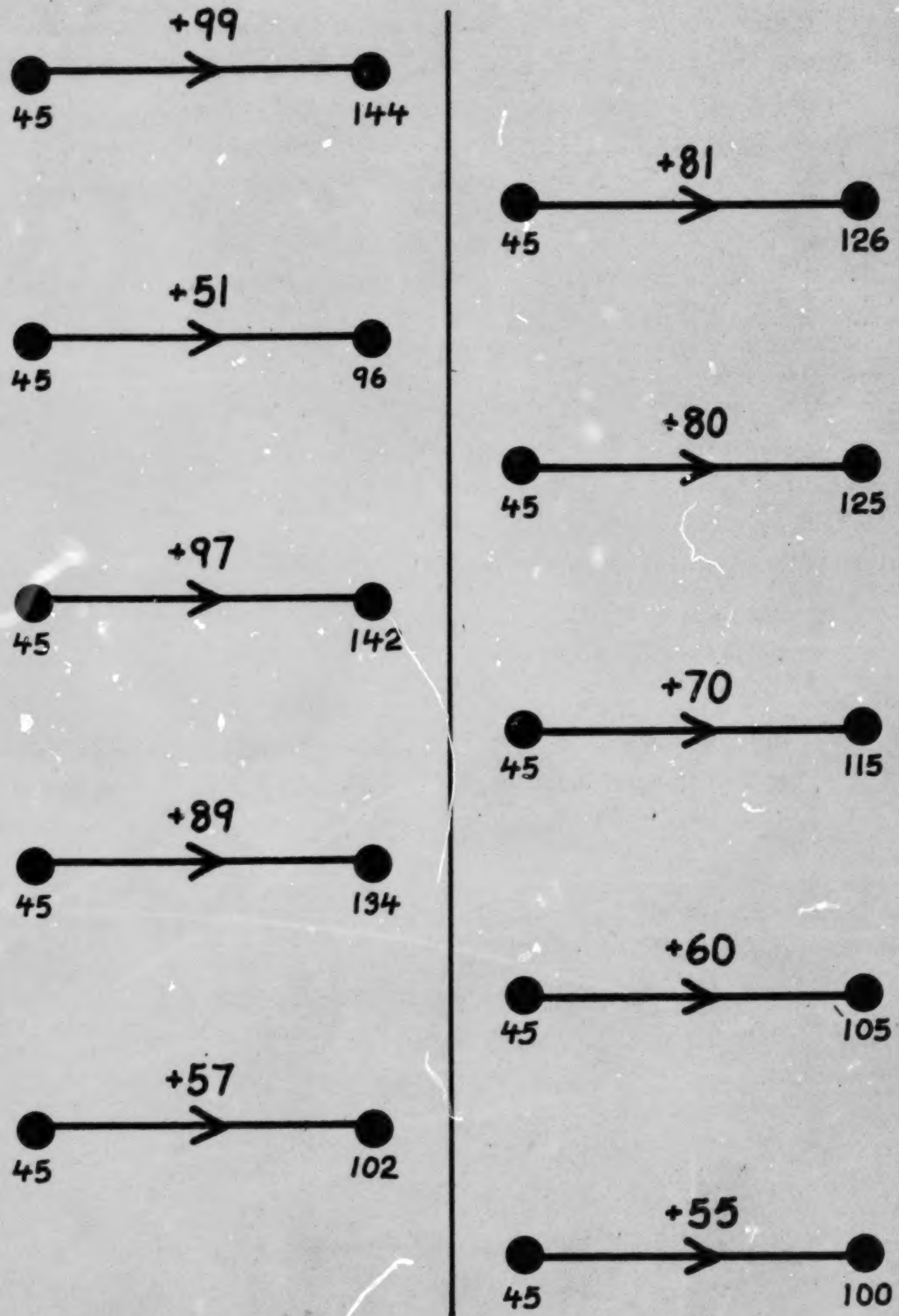




Ralph applies his usual strategy successfully and wins the game with a score of 2. Although Barry is not sure how to exceed the goal in one move, he does know once he passes it how to reach it immediately.

T: Let's play a new game. Display 45 on your hand-calculator. The goal of this game is to reach 100 in ONE move. You will play in turn. The first one to solve the problem will be the winner.

Ten tries are made alternately by Barry and Ralph with the last one a victory for Ralph.


+99
+51
+97
+89
+57
+81
+80
+70
+60
+55

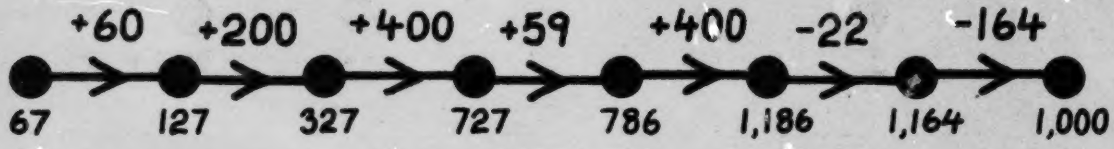
After observing the two boys during this activity and studying the records of their moves, we are inclined to say that each boy was following his own train of thought and was not much influenced by the other player.

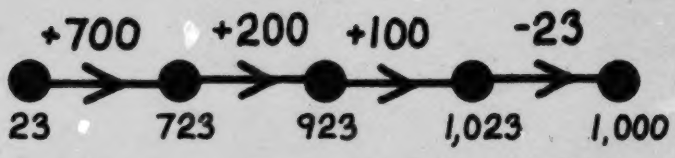
May 6

For administrative reasons, we were unable to arrange any sessions with the boys between March 11 and May 6.

T: Today we will play our hand-calculator games again.
Let's start at 67. You will play in turn, pressing
$\boxed{+}$ $\boxed{\text{any number}}$ $\boxed{=}$ or $\boxed{-}$ $\boxed{\text{any number}}$ $\boxed{=}$
The first player to reach 1,000 will be the winner.

The game is played three times in a row.


+60
+200
+400
+59
+400
-22
-164
67
127
327
727
786
1,186
1,164
1,000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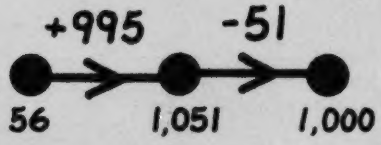
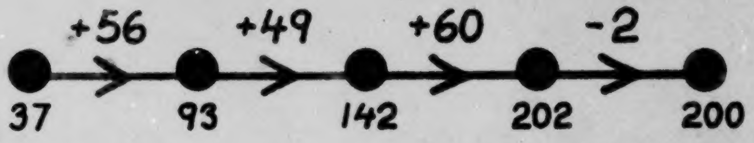
+200
+100
-23
23
723
923
1,023
1,000


+995
-51
56
1,051
1,000

Although the boys are aiming for a larger goal than in the March games, they now reach it in fewer steps and improve each time. Barry's boldness is striking, compared to his earlier caution. In each game his last move confirms that he knows how to reach the goal immediately when he is above it. Neither of the boys seems to anticipate that by going above the goal, he gives his opponent an easy winning move.

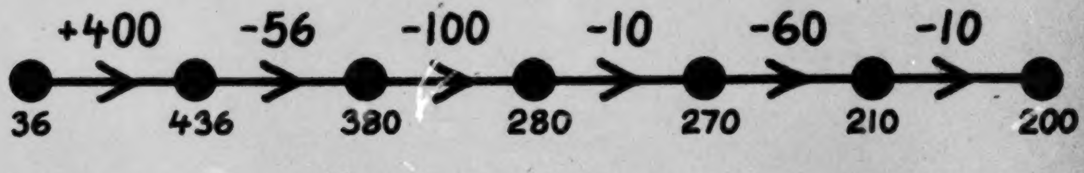
T: Let's change the rule of the game. Starting from a given number, you will have to try to reach 200 in the smallest number of moves. Barry, play first starting at 37.

Barry plays:



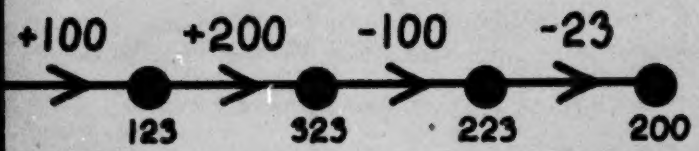
T: Barry's score is 4. It's your turn, Ralph. Start at 36.

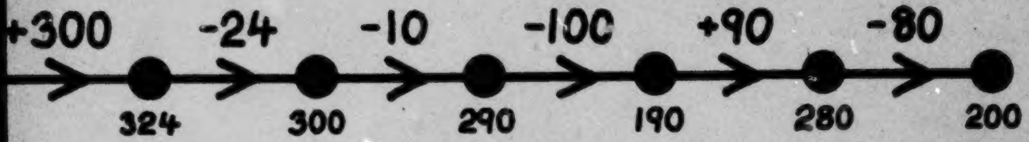
Ralph plays:

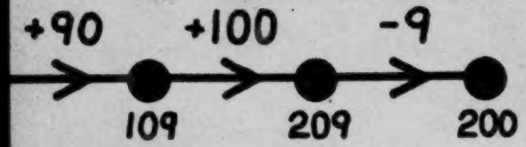


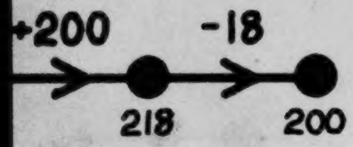
T: Ralph's score is 6.

boy plays two more times.


+100
+200
-100
-23
123
323
223
200


+300
-24
-10
-100
+90
-80
324
300
290
190
280
200


+90
+100
-9
109
209
200

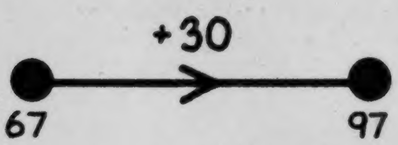

+200
-18
218
200

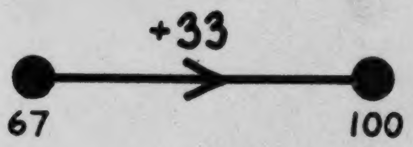
Ralph seems to remember that a first good move is to go above the goal. But in the first two games he struggles with subtraction problems and scores poorly. After a moment of deep concentration, he eventually wins the last game in two moves.

Barry is more steady in his endeavor but never succeeds in scoring 2.

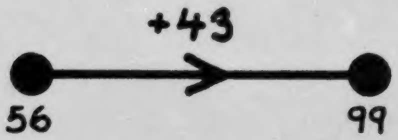
T: Let's try to go from 67 to 100 in one move.

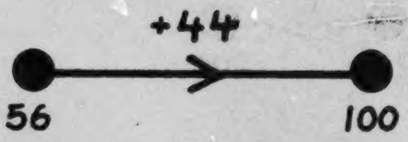
Barry plays first and Ralph wins immediately after.





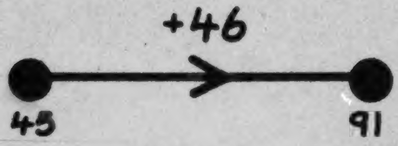
From 56 to 100:

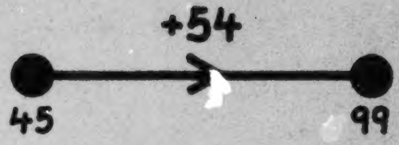


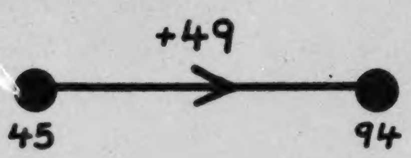


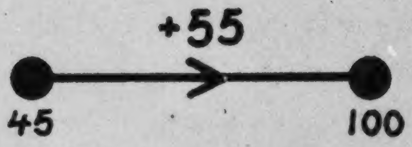
The situation is reversed. Ralph plays first and Barry wins.

From 45 to 100:



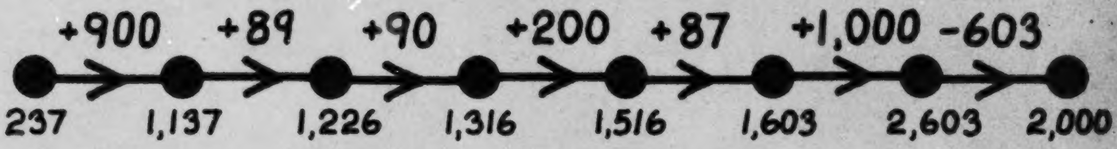






It takes four tries to reach 100 in one move compared to the ten tries in March. Again, the boys do not seem to be influenced by each other's attempts.

At this point the teacher feels that the boys are ready for a big jump and suggests a game from 237 to 2,000. The game is played in its first version.



After the fifth move, 1,603 is on the display. Ralph says "plus ten thousand" but actually presses $\boxed{+}$ $\boxed{1,000}$. He explains that he pressed one, then three zeros. Barry smiles, presses $\boxed{-}$ $\boxed{603}$, and wins the game.

Most average students from grade 4 or grade 5 would recognize that no version of these so called "hand-calculator games" is really a game, since in every case the first player can win in one move. But our two boys still have a very limited picture of the number line and their numerical skills are poor. So they see the situation from a different point of view and enjoy the three activities as real games.

The first one is a very free situation which provides an opportunity for the children to explore large numbers without the fear of making a "wrong move". Such free exploration gives them a chance to make personal discoveries about large numbers and to establish a friendly and trustful relationship with their hand-calculators.

By choosing larger and larger numbers as goals of the game, the teacher indirectly communicates the message that the students are not mathematically illiterate and that she believes in their ability to solve serious problems.

In the second activity, there is still great freedom but some limitations have been introduced to encourage the students to begin developing strategies. Their frequent scores of 2 show they have made a successful first step in this direction.

After some exposure to the first two activities, neither of the boys suggests the possibility of reaching the goal in one move. So the teacher introduces a third game that is, at their own level, an estimation exercise. The laborious ten tries which were necessary for Barry and Ralph to go from 45 to 100 clearly show their weakness in computation. But the real improvement of their estimates during the second session as compared to the first indicates the therapeutic value of the three games.

Even when he lost -- and it happened many times because his bold moves and his natural impulsiveness gave easy victories to his opponent -- Ralph was smiling and joking. His effervescent personality and willingness to risk error helped Barry to relax and come out of his shell.

We believe that the real teacher in the room was the game. The adult simply chose the game and progressively increased the challenge by changing the goal or suggesting new rules. But she left full responsibility for the moves to the boys. The teacher and the observer, who have had many opportunities to observe the two boys in other circumstances -- collective lessons, field trips, playground -- see a striking similarity between the personality of each boy and the profile of the arrow pictures of their games. Ralph's buoyant personality, his surprisingly good intuition, and his boldness are visible in the red arrows. Barry's lack of confidence and more submissive nature appear in the blue arrows. We agree with Virginia M. Axline who writes: "When he (the child) plays freely and without direction, he is expressing his personality. He is experiencing a period of independent thought and action. He is releasing the feelings and attitudes that have been pushing to get out into the open." ([A] p. 22)

1.3 Games With Decimals

Considering the progress of the slow learners, many of the collective lessons about decimals in grades 4 and 5 were both disappointing and frustrating to the teacher and the CSMP development staff. In spite of the concrete support of the terminology of money, the decimals did not seem to have any meaning or any appeal for that group of students. As they might say, "The decimals blew their minds." Something had to be done in the math play therapy sessions.

In the middle of the year, the teacher decided to try a version of the hand-calculator game involving decimals. The results were striking. The students not only responded to the game with an enthusiasm that the teacher did not expect, but they made quick progress and soon become comfortable with the new numbers. Here are some excerpts from these sessions.

February 24

Four of our "fastest" slow learners are present. We will call this group "A".

T: Today we will play a new game with the hand-calculator.

Let's start at 2.4. You can press

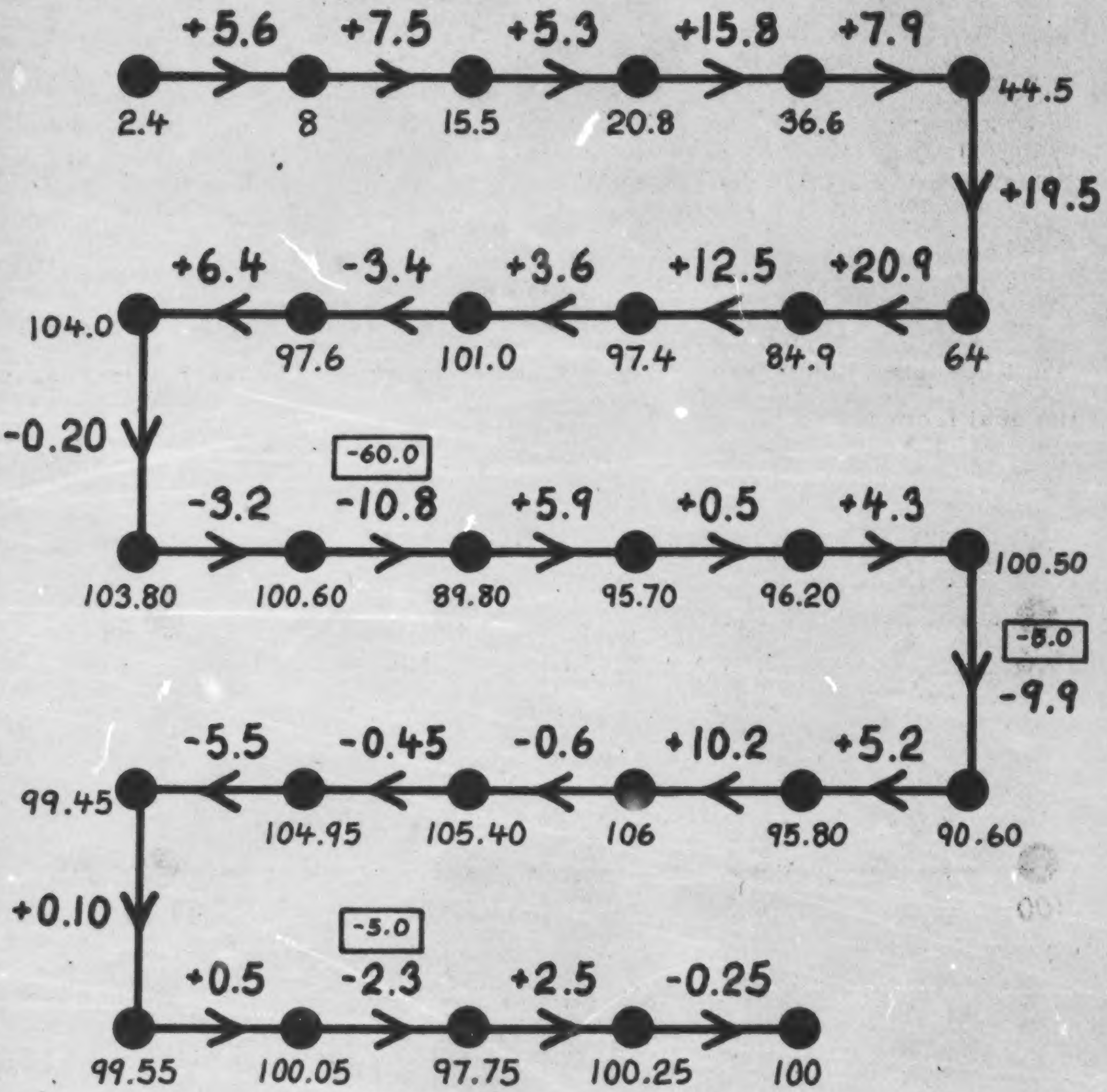
$\boxed{+}$

or $\boxed{\text{any number which is not a WHOLE number}}$ $\boxed{=}$

$\boxed{-}$

You will take turns. The first player to get 100 will be the winner.

The first game uses 28 moves!


+5.6
+7.5
+5.3
+15.8
+7.9
2.4
8
15.5
20.8
36.6
44.5
+19.5
+6.4
-3.4
+3.6
+12.5
+20.9
104.0
97.6
101.0
97.4
84.9
64
-0.20
-60.0
-3.2
-10.8
+5.9
+0.5
+4.3
103.80
100.60
89.80
95.70
96.20
100.50
-5.0
-9.9
-5.5
-0.45
-0.6
+10.2
+5.2
99.45
104.95
105.40
106
95.80
90.60
+0.10
-5.0
+0.5
-2.3
+2.5
-0.25
99.55
100.05
97.75
100.25
100

Three times during the game (see the blue boxes on the picture), the teacher has to refuse a move which is forbidden by the rules of the game ($\boxed{+}$ or $\boxed{-}$ $\boxed{\text{any number which is not a WHOLE number}}$ $\boxed{=}$). These forbidden moves indicate the students' thinking.

$\boxed{-\ 60.0}$ when 100.60 is displayed.

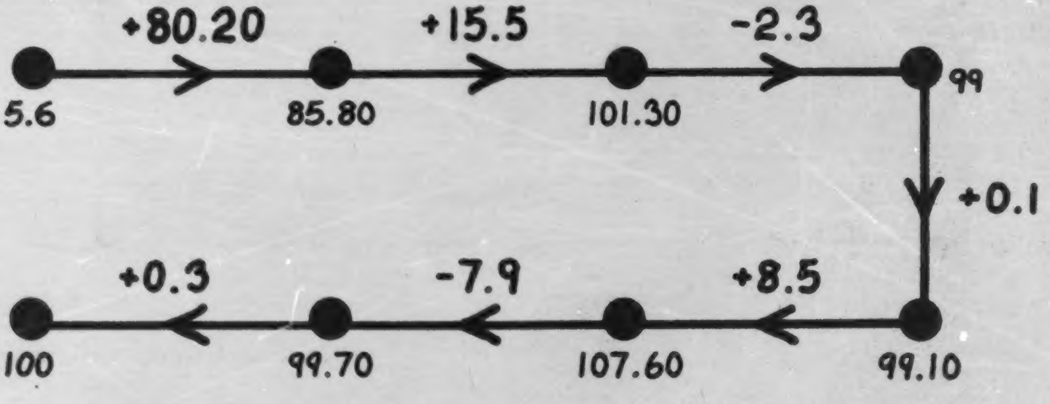
$\boxed{-\ \ 5.0}$ when 100.50 is displayed.

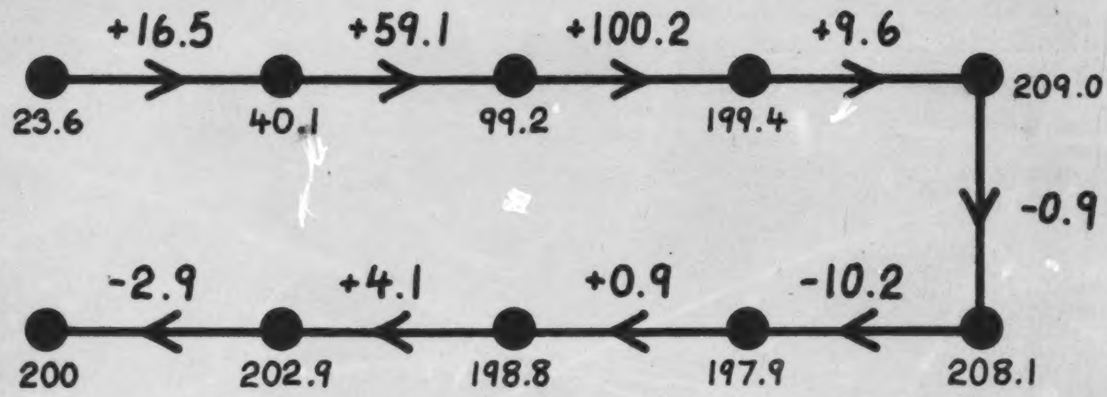
$\boxed{-\ \ 5.0}$ when 100.05 is displayed.

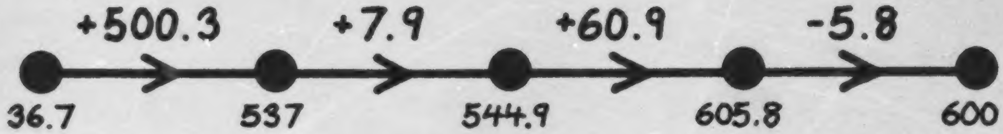
The students ask to play a second game but do not have time to finish it

March 3

One week later, group A plays three games in a row. The teacher increases the goal from 100 to 200 and eventually to 600.



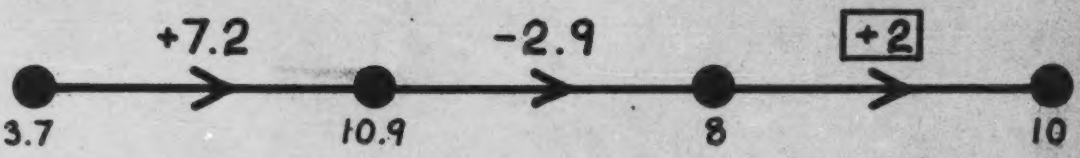

+16.5
+59.1
+100.2
+9.6
23.6
40.1
99.2
199.4
209.0
-0.9
-2.9
+4.1
+0.9
-10.2
200
202.9
198.8
197.9
208.1


+500.3
+7.9
+60.9
-5.8
36.7
537
544.9
605.8
600

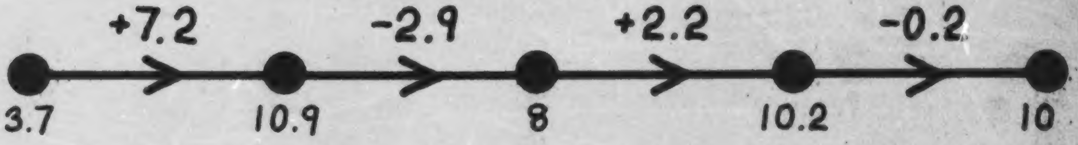
The pictures speak for themselves. Although the goals are more distant, the games are completed with fewer moves than one week earlier. The players are more resolute. No forbidden move has to be rejected. More than one student has found how to reach the goal when the number is above it or in its very close neighborhood.

T: Let's change the rules of the game. Slim will play all by himself, starting at 3.7. He will try to reach 10 with the smallest number of moves. The other students will follow Slim's game on their own hand-calculators.

Slim plays:

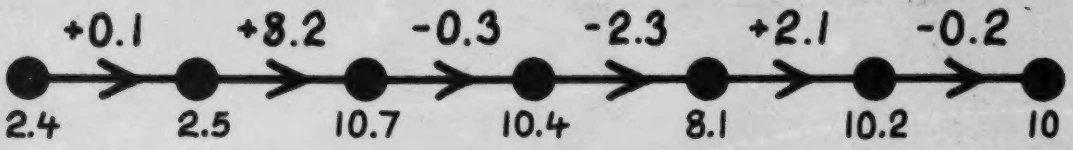


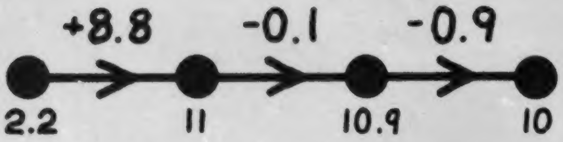
His last move has to be rejected according to the rules of the game. He substitutes a decimal number, plays again, and reachs 10 after the fourth move:

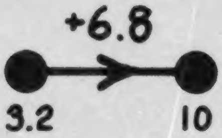


T: Your score is 4.

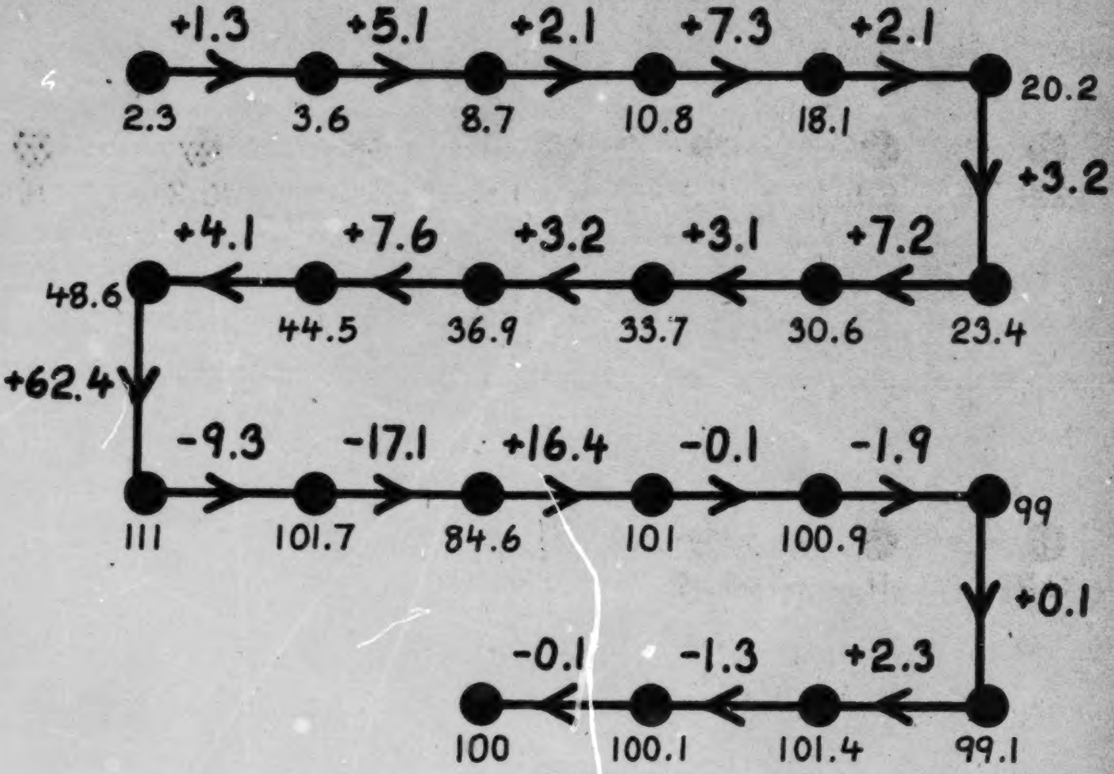
The other players successively make scores of 6, 3, and 1.

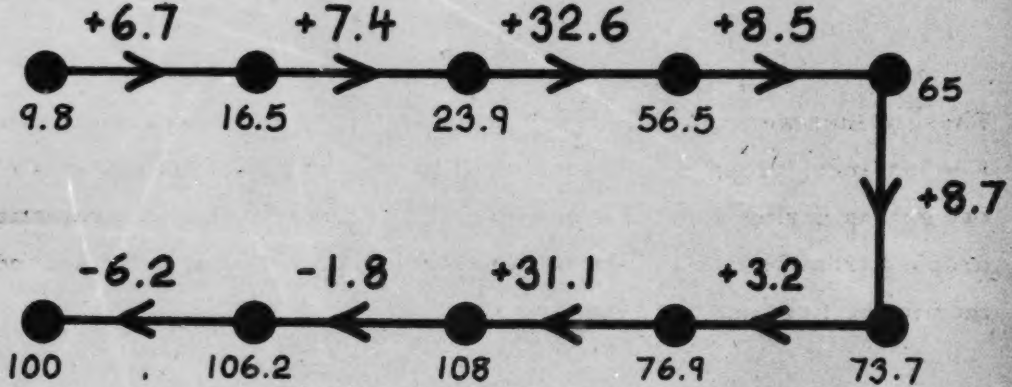

+0.1
+8.2
-0.3
-2.3
+2.1
-0.2
2.4
2.5
10.7
10.4
8.1
10.2
10


+8.8
-0.1
-0.9
2.2
11
10.9
10



These games were played with other groups of students with similar results. For instance, group B which includes some of our weakest students played two games during their first session. The number of moves dramatically dropped from 21 to 9 and the players were less tentative in the second game than in the first one.

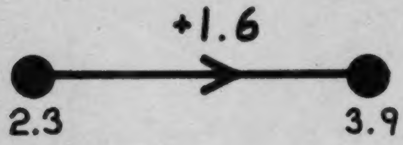

+1.3
+5.1
+2.1
+7.3
+2.1
2.3
3.6
8.7
10.8
18.1
20.2
+3.2
+4.1
+7.6
+3.2
+3.1
+7.2
48.6
44.5
36.9
33.7
30.6
23.4
+62.4
-9.3
-17.1
+16.4
-0.1
-1.9
111
101.7
84.6
101
100.9
99
+0.1
-0.1
-1.3
+2.3
100
100.1
101.4
99.1

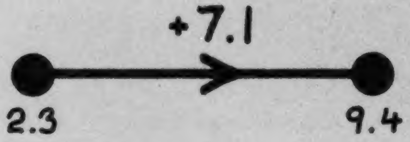

+6.7
+7.4
+32.6
+8.5
9.8
16.5
23.9
56.5
65
+8.7
-6.2
-1.8
+31.1
+3.2
100
106.2
108
76.9
73.7

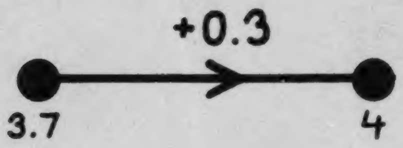
In the second or third session devoted to this topic, depending on the ability of the group, the teacher presents a third version of the game.

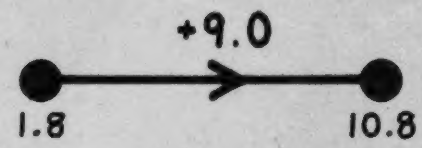
T: We will have a contest between Max and Jack.
I will alternate giving you a starting number and you will have to try to reach 10 in one move. The first player to succeed will be the winner.

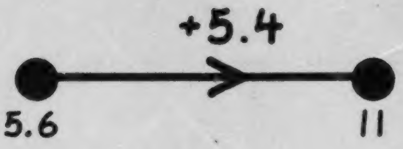
The red arrows represent Max's moves and the blue arrows represent Jack's moves.

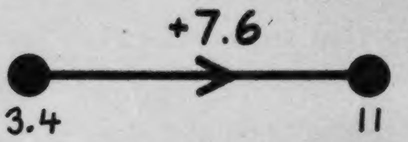


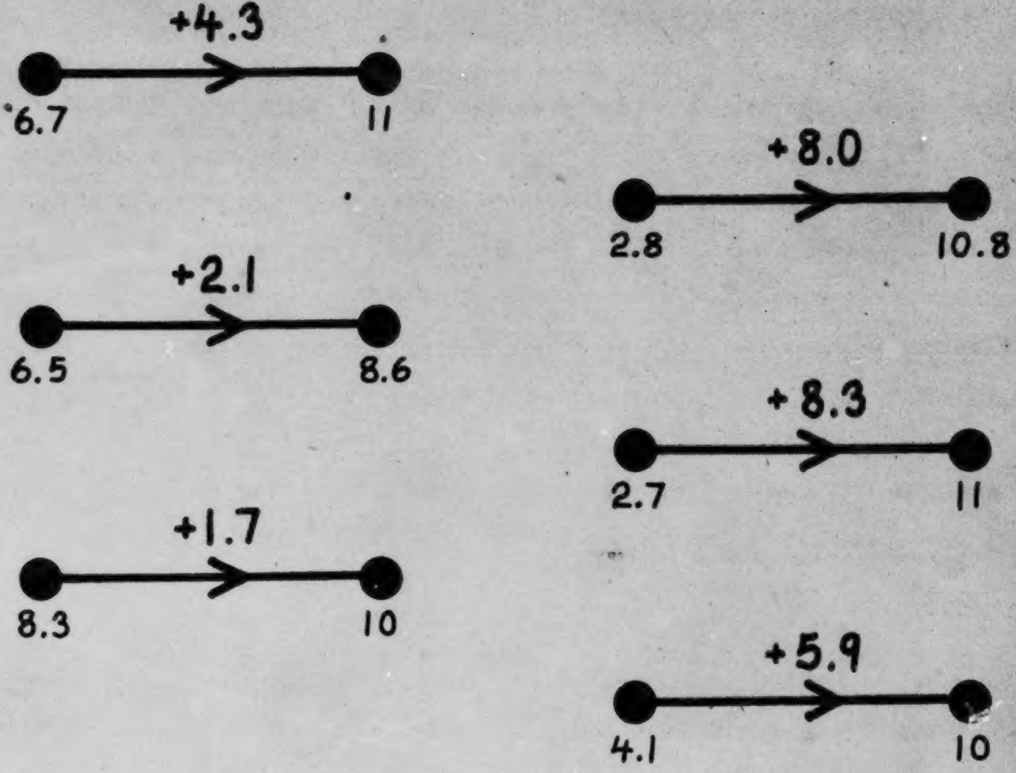












It takes 11 tries before Max, who is the winner, can reach the goal in one move. After Max's victory, the teacher gives a last chance to Jack who then also succeeds.

The teacher was amazed by the positive results of these sessions. She experienced more vividly than before the actual condition of her slow students in the collective classroom situation. Before they have time to become even slightly familiar with a concept, they are overwhelmed by a new one. To make matters worse, each time a new concept is presented other students grasp it quickly. The brisk pace followed by the class does not give the slow learners a chance to take part in the collective discovery. They generally react in a defensive way by pretending not to be interested, or by disturbing the class as if to say, "I'm here too!"

This is what happened in many lessons devoted to decimals. It is particularly interesting to observe that the concrete support of the terminology of money which was helpful for other students, did not seem to have any significance for the slow students. On the other hand, the games which were introduced with the help of the hand-calculator improved the situation. In the context of free play, even the weakest students were fascinated by the decimals. They were eager to try moves and to be challenged.

Once again, a simple game was the best and only teacher. The interaction of the students' personalities and their diverse responses to the game stimulated the creative thinking that was needed.

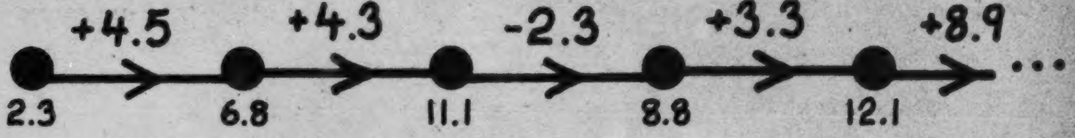
The game using decimals gave rise to strong, we might even say explosive, reactions from three different children. We will report our observations in each of these individual cases.

Nick is a quiet and serious boy, a frequent daydreamer and a poor reader. We had only two students on the day of Nick's first exposure to the game, Nick and Nadine. Nadine is a gentle and quiet girl who succeeds more often than Nick in computation, but who is not viewed by Nick as a rival.

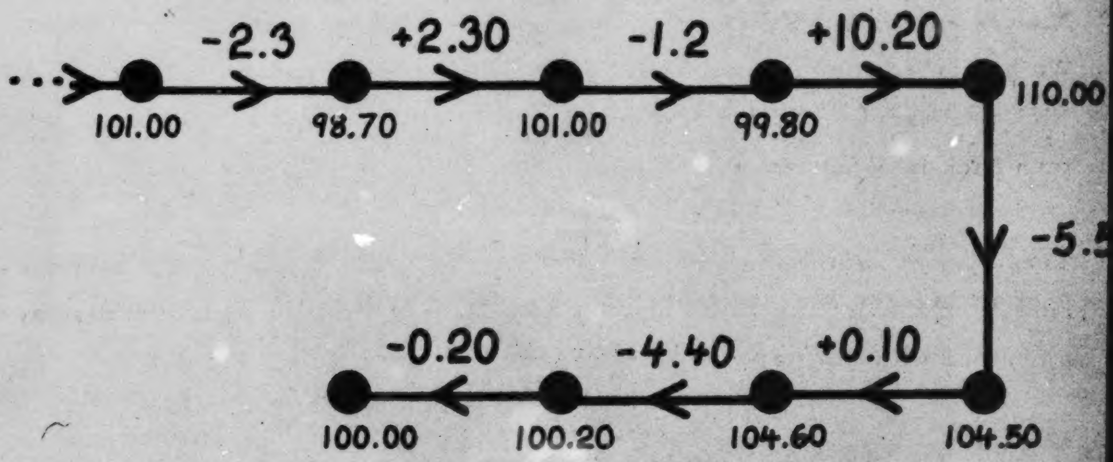
They play two games in a row. Nick is tense because he does not have good control of the decimal point button on his hand-calculator and often displays a number which is not what he had intended.

The red arrows represent Nick's moves and the blue arrows represent Nadine's moves.

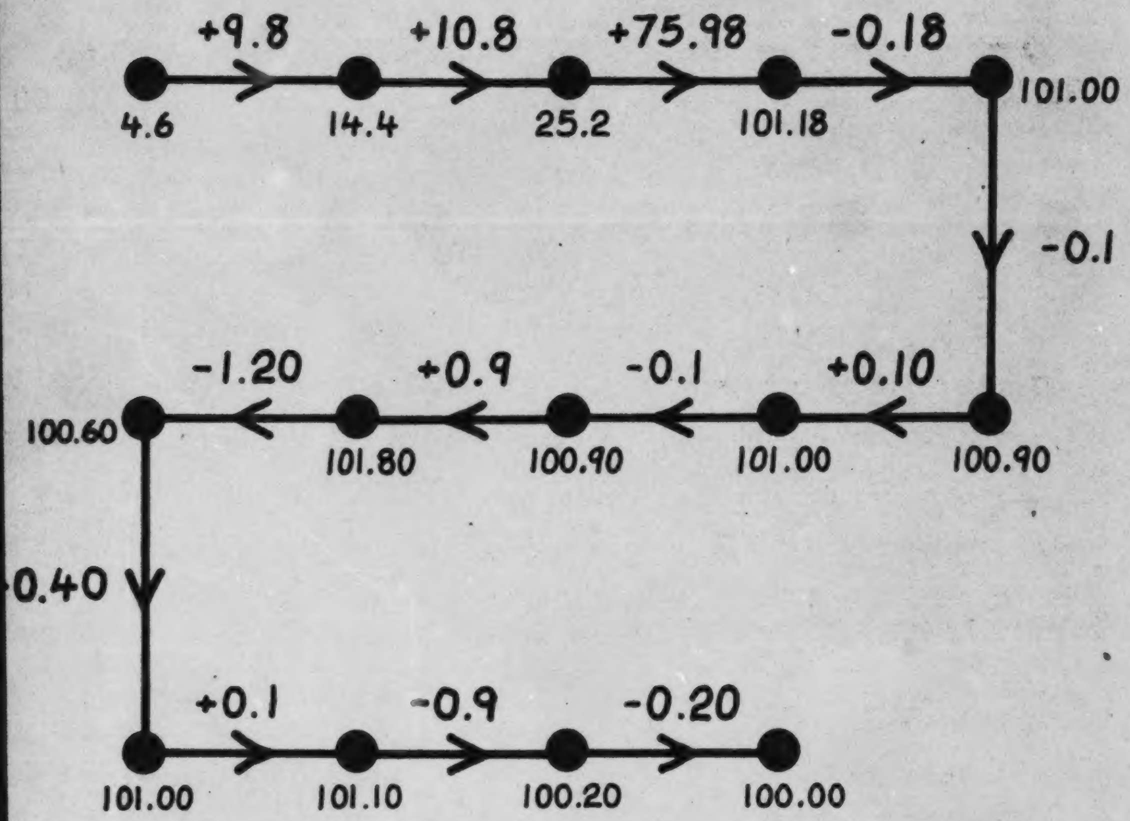
First game:


+4.5
+4.3
-2.3
+3.3
+8.9
2.3
6.8
11.1
8.8
12.1

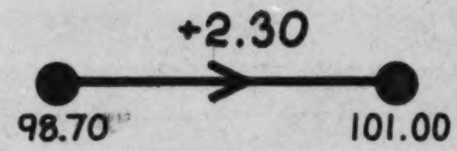
The game goes on for about ten moves, but we have no records of this sequence because the observer was called out of the room for a few minutes. The last eight moves of the game were recorded and indicate that in the end the victory was Nick's.

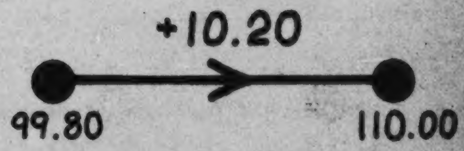

-2.3
+2.30
-1.2
+10.20
101.00
98.70
101.00
99.80
110.00
-5.5
-0.20
-4.40
+0.10
100.00
100.20
104.60
104.50

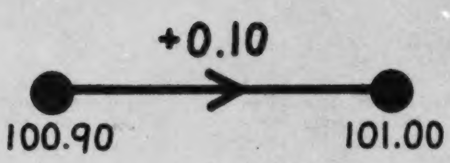
Second game:

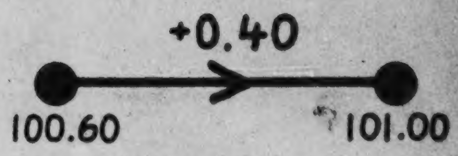

+9.8
+10.8
+75.98
-0.18
4.6
14.4
25.2
101.18
101.00
-0.1
-1.20
+0.9
-0.1
+0.10
100.60
101.80
100.90
101.00
100.90
0.40
+0.1
-0.9
-0.20
101.00
101.10
100.20
100.00

Even though the record of the first game is incomplete, the arrow pictures of the two games seem to show that Nick does not play at random and is on the verge of discovering a strategy to reach the goal. We have extracted from the two games four moves that indicate the direction of his thinking and explain his frustration.


+2.30
98.70
101.00


+10.20
99.80
110.00

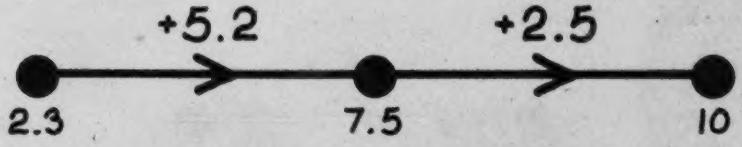

+0.10
100.90
101.00


+0.40
100.60
101.00

After two games, the teacher suggests:

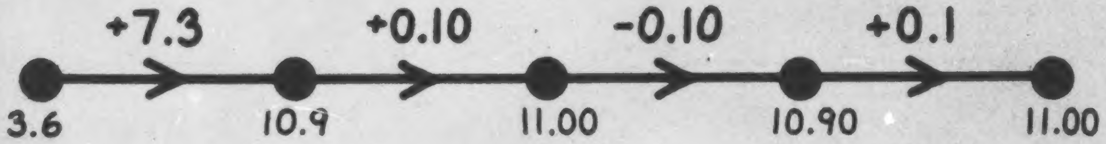
T: Let's change the rules of the game. Starting from a given number, you will have to try to reach 10 in the smallest number of moves.

Nadine plays first and makes a score of 2.


+5.2
+2.5
2.3
7.5
10

During Nick's turn his body becomes increasingly tense. His face reflects the deep strain of an inner struggle which is obviously painful. He suffers because he is unable to organize his thinking. He intuitively senses a correct strategy, but he cannot put together the pieces in order to have a well defined mental picture as he wishes.

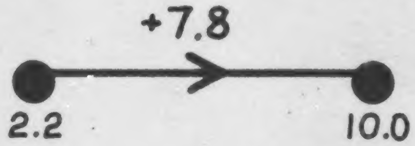
Nick plays:



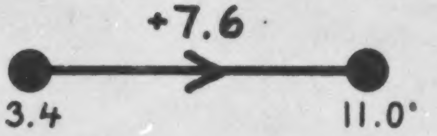
After the fourth move he stops, eyes brimming with tears, and wrestles to gain control of his emotions. It is obvious that he is determined to summon new strength to deal with this most serious problem and to master it.

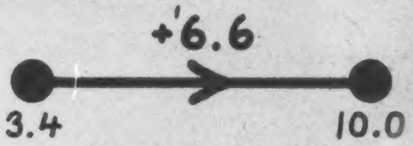
T: Nick, it's okay to cry and to let yourself rest for a moment. Nadine can take a turn.

The tension gradually drains out of Nick's body and he relaxes while Nadine makes a successful try.

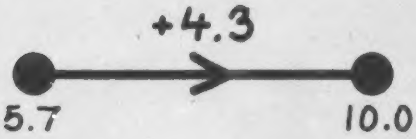


Then he is ready to play again and is not discouraged by his first attempt. Instead, when he reaches 11, he says with a trace of excitement, "May I start over?" The teacher agrees, and this time he achieves his goal!





The teacher then gives him a new starting number and he reaches the goal in one move. His face and body radiate his intense satisfaction at breaking down another barrier to his understanding.

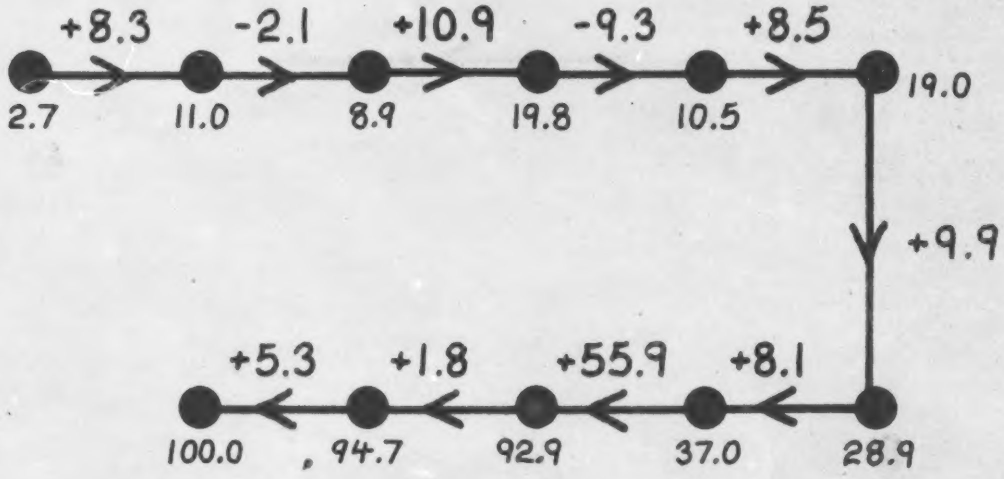


Harry is a tall, husky boy whose physical appearance is almost that of an adult. Is this the reason why he, more than many other students, feels ashamed of his difficulty in following the brisk pace of the collective lessons? He never volunteers an answer and when called on, he generally refuses to respond even to a simple question. He prefers to remain silent rather than risk giving an incorrect answer, which might bring ridicule from his peers.

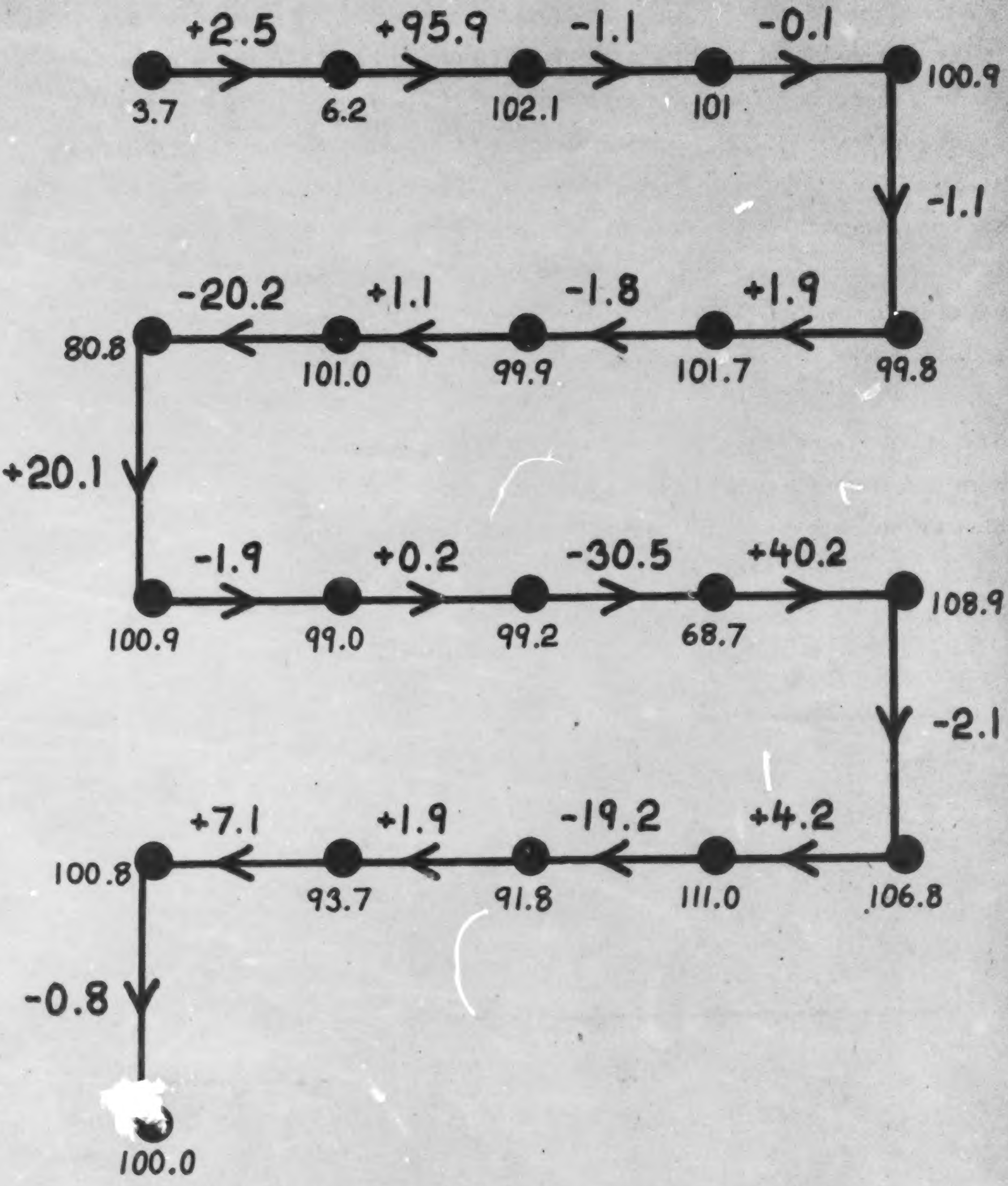
In the small groups, Harry is relaxed and very proud when he can play a leading role. But his behavior during the games often causes problems because he thinks too long before making a decision. When he has not found a strategy for winning, either he makes an overcautious move in order not to lose or he makes an exaggerated move which does not give him any chance to win but also deprives his opponent of the chance to win. During his first exposure to the hand-calculator game using decimals, he often acted in one of these ways.

The next illustrations are from the record of two games that Harry played with three other boys. The red arrows represent Harry's moves and the blue arrows represent the moves of the other players.

First game:

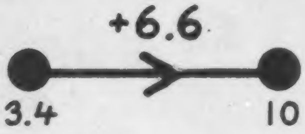


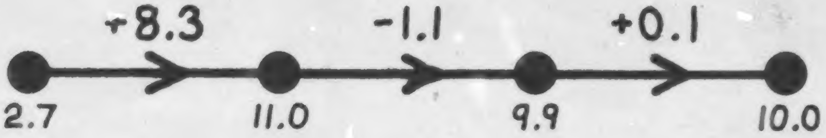
Second game:

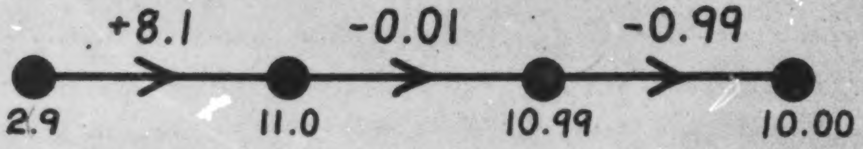

+2.5
+95.9
-1.1
-0.1
3.7
6.2
102.1
101
100.9
-1.1
-20.2
+1.1
-1.8
+1.9
80.8
101.0
99.9
101.7
99.8
+20.1
-1.9
+0.2
-30.5
+40.2
100.9
99.0
99.2
68.7
108.9
-2.1
+7.1
+1.9
-19.2
+4.2
100.8
93.7
91.8
111.0
106.8
-0.8
100.0

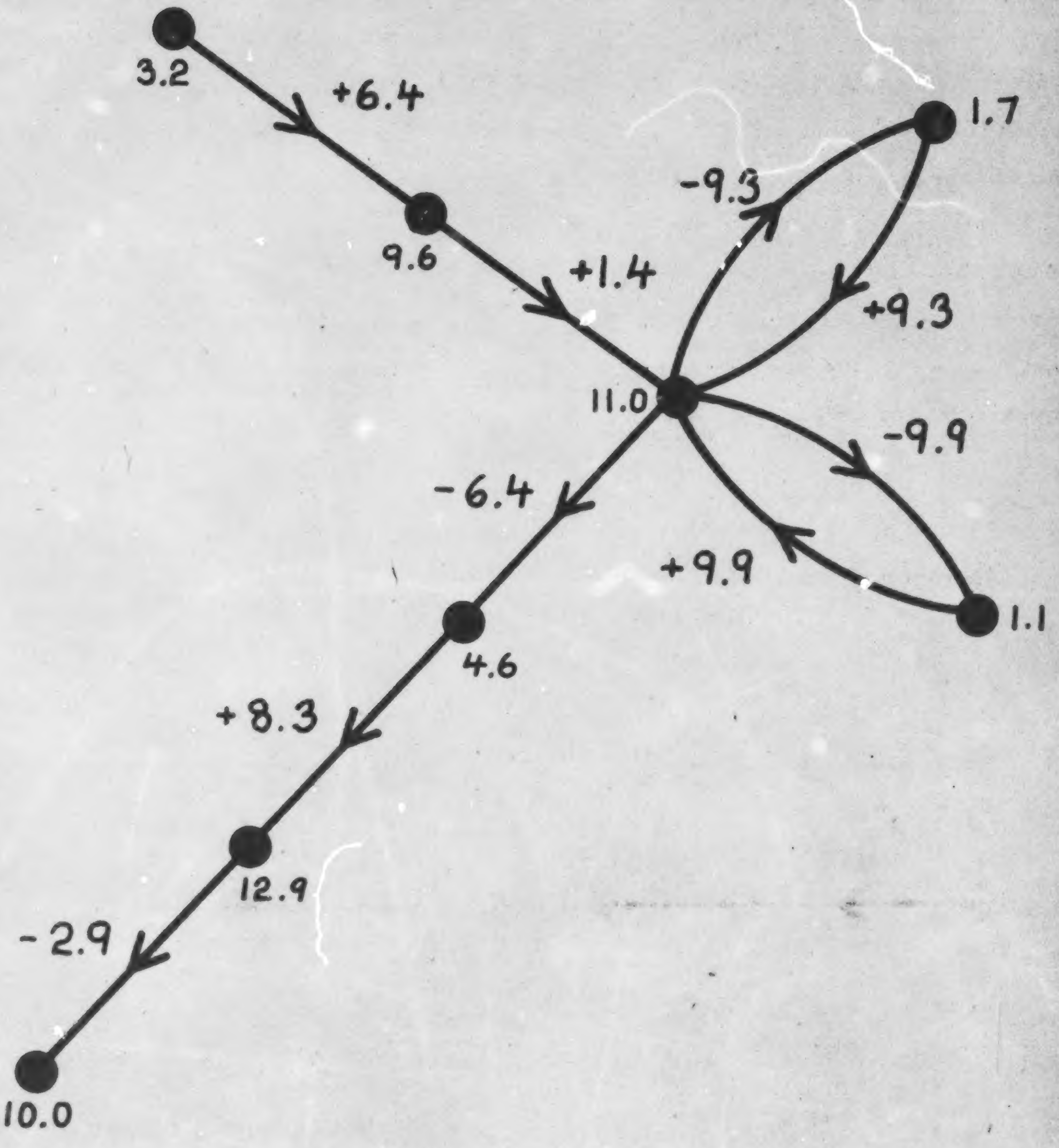
By glancing at the arrow pictures, the reader might conclude that Harry really has a weak understanding of computation with decimals. All his moves in the first game and his first move in the second game are very tentative. In the second game, his fourth move shows that he does not know -- or that he is too tense to know -- how to go from 99.2 to 100. His strange -30.5 move gives him no chance to win, but it places the next player in a more difficult position. In the other two cases, he has to play when the numbers 101.0 and 110.0 are displayed. In accordance with the rules of the game, he cannot win because he cannot subtract a whole number. But he anticipates that his -20.2 and -19.2 moves will give the next player only a small chance to win.

After these two games, the teacher suggests a second version of the game. Each player has to try to go from a given starting number to 10 in the smallest number of moves. Harry plays first and makes a score of 1. The other boys score respectively 3, 3, and 9.






+8.1
-0.01
-0.99
2.9
11.0
10.99
10.00

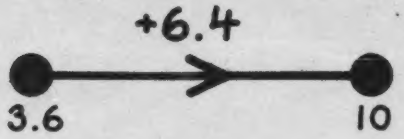

3.2
+6.4
9.6
+1.4
1.7
-9.3
+9.3
11.0
-9.9
+9.9
1.1
-6.4
4.6
+8.3
12.9
-2.9
10.0

Because the teacher suspected that Harry had played well below his actual ability and that his deep anxiety had prevented him from making better plays, she offered to let him come alone and be challenged with more difficult problems. He immediately accepted.

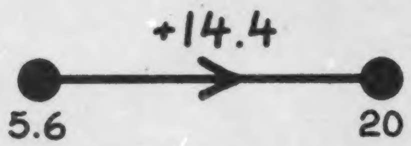
The following dialogue is from the transcript of that session which occurred two days later and took about twenty minutes.

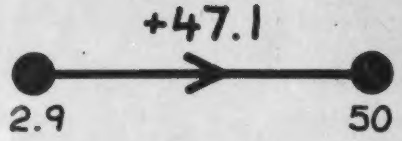
T: Start from 3.6 and try to get 10 in one move.

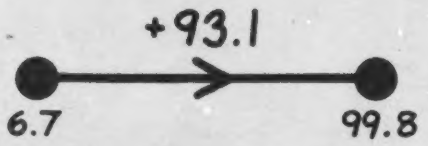
With just one try, Harry succeeds:



The teacher increases the goal successively to 20, 50, and 100.



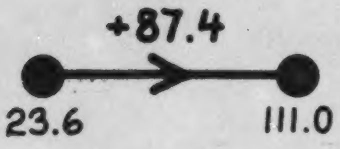


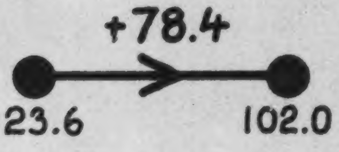


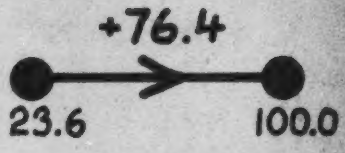
H: Oh, it should be 93.3.

T: Now the game is too simple for you. I have to make it harder. Try to go from 23.6 to 100 in one move.

After three tries, Harry succeeds:

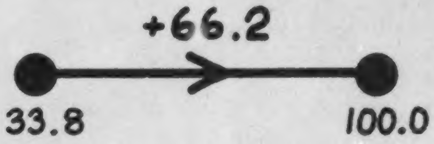

+87.4
23.6
111.0


+78.4
23.6
102.0

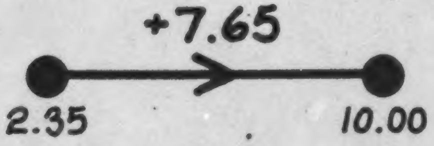

+76.4
23.6
100.0

T: Try to go from 33.8 to 100 in one move.

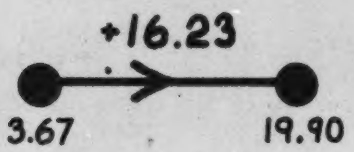
One try and Harry succeeds:

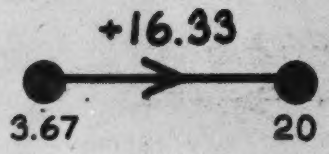

+66.2
33.8
100.0

T: Here are some more difficult problems. Try to go from 2.35 to 10 in one move.

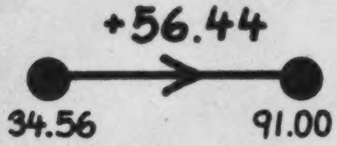

+7.65
2.35
10.00

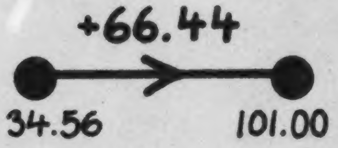
To go from 3.67 to 20, Harry makes two tries:

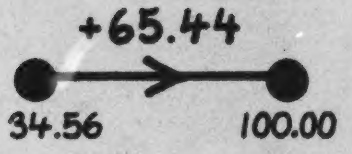

+16.23
3.67
19.90



To go from 34.56 to 100 takes three tries:

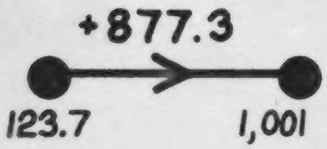

+56.44
34.56
91.00

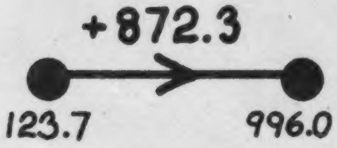

+66.44
34.56
101.00

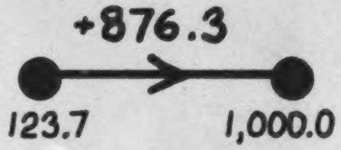

+65.44
34.56
100.00

After the second try, Harry says: "Oh, I see my mistake!"

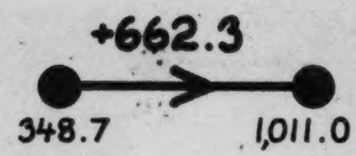
To go from 123.7 to 1,000 takes three tries:

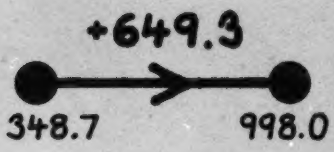

+877.3
123.7
1,001

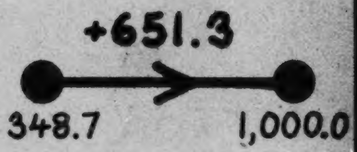

+872.3
123.7
996.0


+876.3
123.7
1,000.0

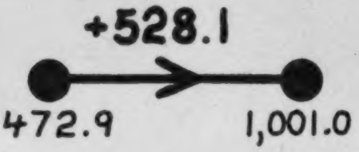
To go from 348.7 to 1,000 takes three tries:

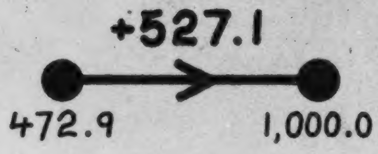

+662.3
348.7
1,011.0


+649.3
348.7
998.0

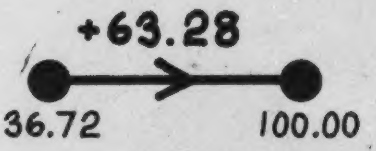

+651.3
348.7
1,000.0

To go from 472.9 to 1,000 takes two tries:


+528.1
472.9
1,001.0


+527.1
472.9
1,000.0

To go from 36.72 to 100 Harry succeeds with one try:


+63.28
36.72
100.00

50

Two factors seem to have played a decisive role in the success of the private session with Harry and may explain the discrepancy noted in his work alone as compared with his work in the group. By offering him a challenge, the teacher showed that she had high expectations for him. When alone, Harry was not distracted by his usual fear of being belittled. Here he was in a position to make full use of his potential, and he did it. His concentration was marvelous to observe. Each time that he did not succeed on the first try, he thought intently for a little while and then with a smile explained why he was wrong and how he could improve his answer.

The teacher was not surprised by what happened. After observing this overly self-conscious boy for nearly two years, she was almost sure that he would reveal his potential if he were placed in the right situation. Is it necessary to say that Harry was overjoyed at the end of the session?

During the two months that followed, Harry did not change his behavior in the classroom. But he eagerly agreed to take part in the small group sessions, something he had refused many times in the past.[*] Furthermore, the quality of his individual work on workbook days improved significantly.

He still needs to be convinced that he can operate as successfully in the whole group as he can by himself. The joy of a more relaxed interaction with his peers remains to be experienced in the future.

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[*] In Chapter 3, pages 113-117, we present excerpts from another session that indicates Harry's capability.

Tina is like a wild animal that has been wounded so many times that it does not trust human beings anymore and desperately rebels against any kind of taming. Furthermore, she does not trust herself. Like Rufus in "A Circle of Children" ([M], p. 140), a serious problem for Tina is that, even in a game, she is "too frightened to commit herself to a choice, terrified that she may make a wrong move." In the collective lessons, she generally does little work and most of the time refuses any help, even from her most sympathetic teachers.

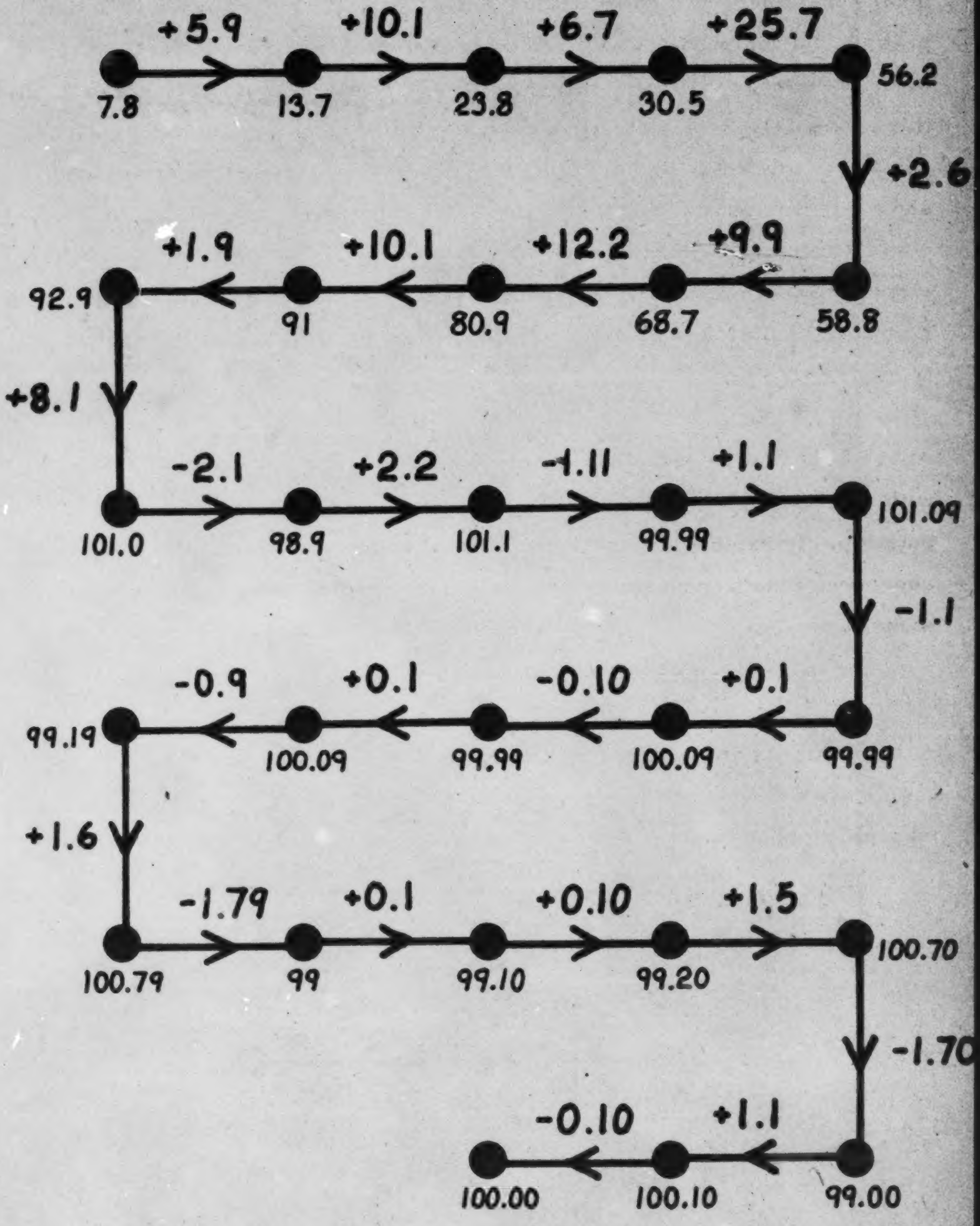
In fourth grade, Tina took part in the small group sessions on a regular basis. During the first months she made a nuisance of herself and distracted the boys in her group. The teacher did not comment negatively about her behavior, but rather tried to make her feel that she was fully accepted as she was. Gradually she became less frightened and a little more trusting. At the end of the year, she was willing to participate in the work of her small group. However, the teacher had to acknowledge that Tina had made no real progress in computation.

After the summer vacation, Tina came back to school with a new determination. During the first weeks, she tried to follow the collective lessons with attention and worked hard all by herself on the easier worksheets. She seemed to be encouraged by her first success. But very soon, for reasons that we cannot completely understand, she returned to her old patterns of behavior. We suspect that her awareness of being far behind other students caused her to conclude that her efforts were not being rewarded. In the collective lessons, she was quickly discouraged and most of the time gave up on a problem.

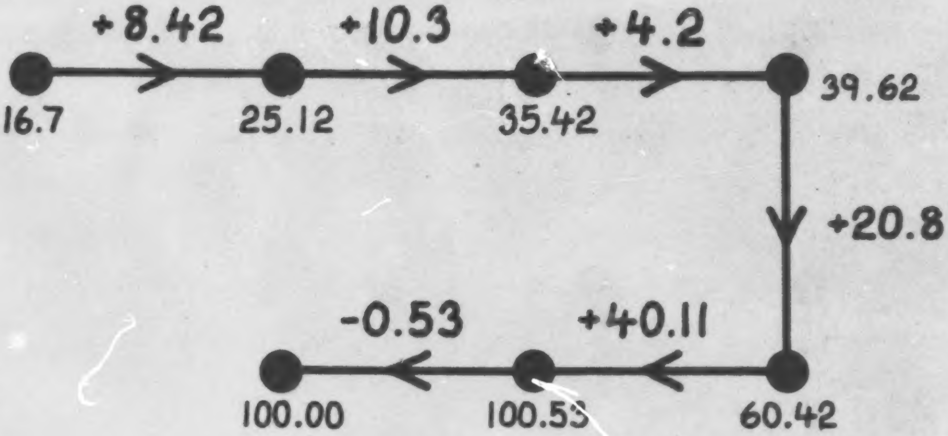
The teacher gave Tina the opportunity to come alone for some sessions, hoping that she would relax without the pressures she felt when with her peers. But the sessions with Tina alone were a complete failure. She was even more tense all by herself than in the presence of other children. After three private sessions, the teacher gave Tina the choice of coming alone, with other students, or not coming at all and made it clear that any choice she made was okay. Her face brightened and without hesitation she chose to come with other children. From this time on, Tina's performance in the small groups improved slowly but steadily. For instance, she became more fluent in the language of the Minicomputer and was reasonably successful in Minicomputer Golf.

At the beginning of March, Tina played the following hand-calculator games with three other students who generally succeed more often than she in computation. It was her second experience with these games.. The red arrows represent Tina's moves and the blue arrows represent the moves of the other students.

First game:


7.8
+5.9
13.7
+10.1
23.8
+6.7
30.5
+25.7
56.2
+2.6
58.8
+9.9
68.7
+12.2
80.9
+10.1
91
+1.9
92.9
+8.1
101.0
-2.1
98.9
+2.2
101.1
-1.11
99.99
+1.1
101.09
-1.1
99.99
+0.1
100.09
-0.10
99.99
+0.1
100.09
-0.9
99.19
+1.6
100.79
-1.79
99
+0.1
99.10
+0.10
99.20
+1.5
100.70
-1.70
99.00
+1.1
100.10
-0.10
100.00

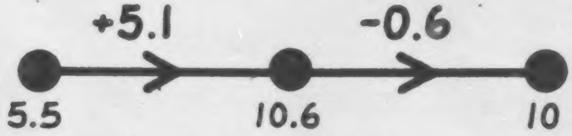
Second game:


+8.42
+10.3
+4.2
16.7
25.12
35.42
39.62
+20.8
-0.53
+40.11
100.00
100.53
60.42

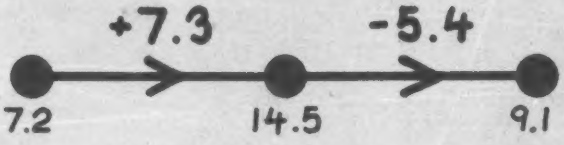
We notice immediately that the number of moves dropped from 27 in the first game to 6 in the second game. In the first game, Tina's last four moves make more sense than many of the moves of her partners and seem to indicate that she is trying to develop a strategy.

In the same session, the teacher changes to the second version of the game. Each player starting from a given number has to try to reach 10 in the smallest number of moves.

Tina plays first:


+5.1
-0.6
5.5
10.6
10

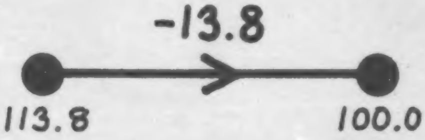
Tina scores 2 and the three other students score respectively 1, 2, and 3. A similar game is played again. Tina stops after her second move . . .



apparently because her -5.4 move (instead of -4.5) does not give her the result that she expected. She refuses to continue.

Two months later[*] Tina is exposed to the same games for the third time. Two other girls are present, and both are definitely better than Tina in computation. Slim is invited to participate but refuses because of Tina's presence.

The group game (first version) is played twice in a row. Tina wins the first game (completed in six moves) with this move:



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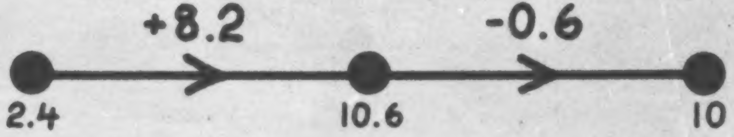
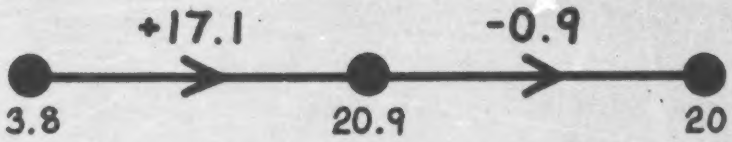
[*] During the two month interval, we had only two sessions which included Tina. They were both devoted to other activities.

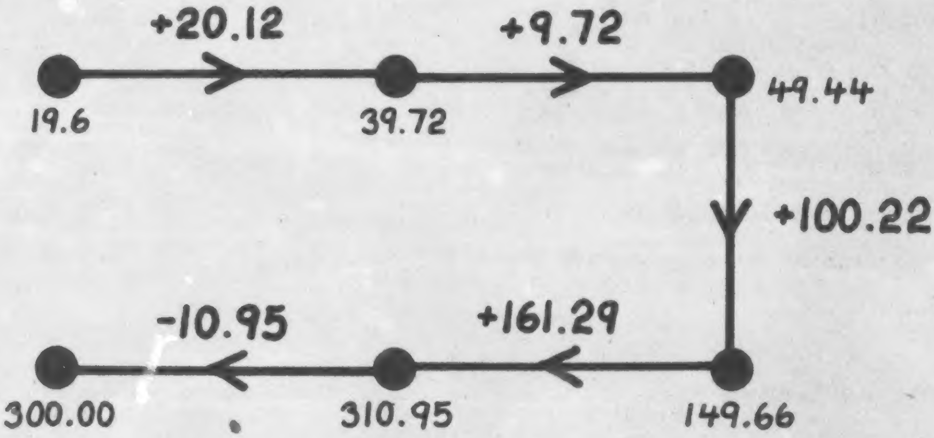
The individual game (second version) is played twice, using goals of 10 and 20 successively In each case, Tina scores 2.


+8.2
-0.6
2.4
10.6
10

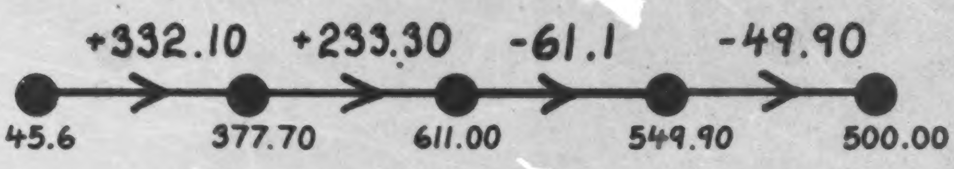

+17.1
-0.9
3.8
20.9
20

One of the other players scores 1 and 2; the third player scores 4 and 5.

Two more group (first version) games are played using respectively goals of 300 and of 500. Tina wins the first game:


+20.12
+9.72
19.6
39.72
49.44
+100.22
-10.95
+161.29
300.00
310.95
149.66

In accordance with the rules of the game, she cannot win on her turn in the second game and her -61.1 move puts the goal in an easy range for the next player.



Perhaps she lost control of the situation because the large move that would have been necessary to make it more difficult for the next player was out of the range of her knowledge. In two years it was the first time that Tina was the star of a group. Still more striking, it was the first time that she played a game in a resolute way, almost anticipating a win.

Immediately after this session, the teacher had the opportunity to observe Tina in a regular math lesson taught by another person. The girl clearly tried to participate actively. She raised her hand many times to respond to questions. She had regained some of the determination that was visible during the first months of the year.

Whether this change will last remains to be seen. In any case, we are convinced that the hand-calculator game using decimals gave Tina a unique opportunity to experience a victory; a victory which began the process of rebuilding the self-confidence she so sorely misses.

2

MINICOMPUTER

2.1 Introduction

The Minicomputer and the hand-calculator are the main tools we have used in math play therapy sessions throughout 1976-77, as they were in the previous year. However, from the very beginning of this year we were in an easier and better situation because all the children, even the very slow, had some fluency with the language of the Minicomputer. Also because the students had gradually become skillful in using the hand-calculator during the first year, they were completely at home with it when the new school term started. It was serving not only to extend their ability to calculate in various ways, but to spark their own mental processes.

The students' fluency with the Minicomputer language was developed in a very natural way by playing games, particularly Minicomputer Golf. As they played they began to develop and expand their own comprehension of numbers and of what happens to a number on the Minicomputer when a checker is moved about within one board or from board to board. By the end of the school year, they were even beginning to understand what happens when a negative checker is used. In addition, the Golf Game provided a strong natural motivation for them to develop skills in estimation and in so doing, clarified and extended their mental picture of the number line. Through their first tentative ventures with negative checkers, they glimpsed an intimation of that part of the number line which is quite often out of the range of comprehension for such children.

A primitive concept of strategical thinking in a numerical context was a natural outgrowth of these experiences. The students now had an alternative to playing blindly; they began to select a move after seeing ahead, perhaps one or two steps, the consequences of such a move.

Early in the fall a happy event occurred that was to play a key role in the subsequent work of the small groups with the Minicomputer. In the context of a detective story[*] taught collectively, the teacher introduced a weighted checker to represent a dime and made the encouraging discovery that even the weakest students quickly understood its significance. She sensed that if she expanded this idea she might have a new visual symbol which would illuminate multiplication for these children in a way that the usual symbols and words did not. So she joined to the original ⑩-checker a set of weighted checkers from ② to ⑨ and gradually developed two kinds of activities. The first was a succession of puzzles using one or a combination of checkers on the Minicomputer in number recognition exercises. The second was a new version of Minicomputer Golf which incorporated some of these new checkers into the game. As the year progressed, not only did the students make a big jump in their grasp of multiplication, but they began to memorize some of the multiplication facts, an accomplishment which had so far eluded them.

2.2 The First Five Minutes

At the beginning of the school year (fall months), almost every session that focused on the Minicomputer began with approximately five minutes of silent communication between the students, the Minicomputer, and the hand-calculators.

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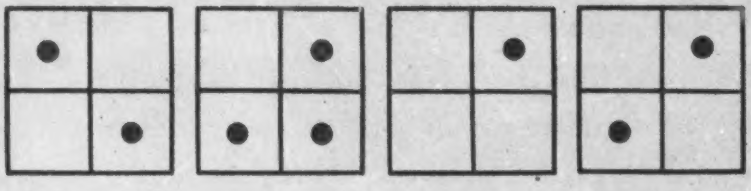
[*] See the lessons on the story-workbook "Clinton Street" in CSMP Mathematics for the Intermediate Grades, Part II, The Workbook Strand, pp. 55-85.

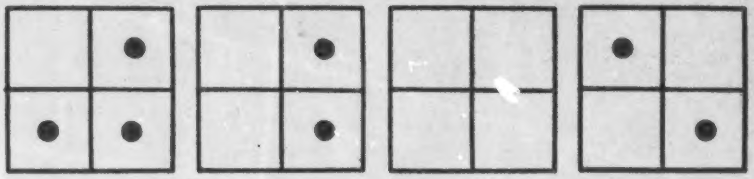
This configuration is on the Minicomputer.



T: Display this number on your hand-calculator. (Answer: 9, 746)

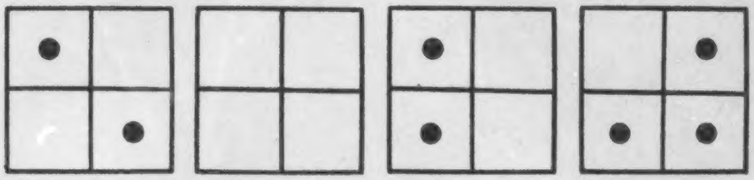
A glance at each display quickly lets the teacher know which students are successful. If any are in trouble she simply has them read the number aloud, then display it.

The teacher presents the following configuration to give the students experience with the empty board, which is a disturbing element for some of them.



T: Put this number on the display of your hand-calculator.
(Answer: 7, 509)

She takes the students a step further with this configuration.



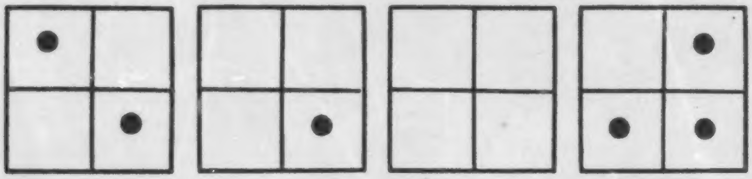
T: Put this number on the display of your hand-calculator.
(Answer: 9, 107)

This time the number is not presented in the standard way. Thus the students might envision the forward play, "80 + 20 = 100", and picture the standard configuration in order to display the number correctly. If some students are not correct, the teacher permits more time for them to try again. The friendly hand-calculator leaves no trace of mistakes. A push of a button quickly allows a fresh start.. If anyone is still unsuccessful, the teacher asks:

T: Who can show the same number with one less checker?

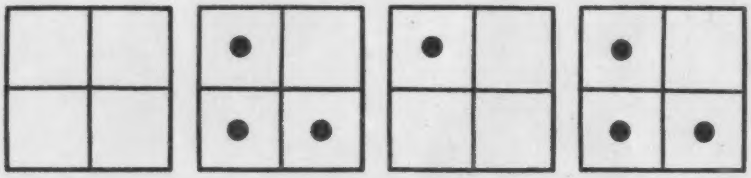
She selects a student who is having difficulty to make the forward move.

80 + 20 = 100



As soon as the number is presented in the standard way, the students succeed in displaying it correctly.

The teacher presents a more difficult situation.



The students mentally must make two forward plays in order to picture the standard configuration for the number. (Answer: 1,191) Time after time in such situations, long moments of deep concentration occur. The thinking is so intense that one is caught up in the mood of suspense and subdued excitement. No doubt children's natural attraction to codes and puzzles partially explains why the students are so captivated. But we think there is a more important reason.

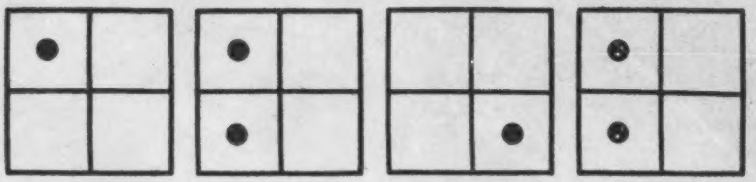
In contrast to many of these children's learning experiences, here they see spread before them an intriguing and complex situation that will not disappear before they even begin to absorb it. They have the time to carry on a personal dialogue with themselves and have the added encouragement of the hand-calculator. They can experiment with mental plays knowing that if they are in error, a push of a button will wipe the slate clean. They are free to try many approaches at a pace that is comfortable to them. Under these circumstances, they invariably disprove the theory that "weak students are incapable of intense, focused thinking."

If students are not successful in their experimentation with the situation of the preceeding illustration, the teacher asks:

T: Who can show the same number with one less checker?

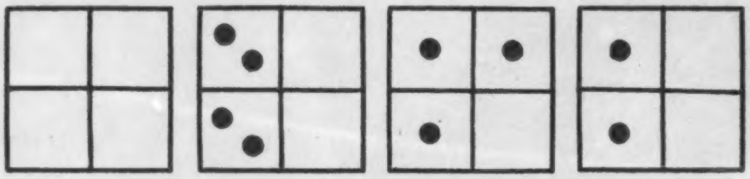
A student is asked to make only one of the forward plays (8 + 2 = 10 or 800 + 200 = 1,000). Then everyone has a new chance to display the number correctly. A student who is still troubled is asked to make a second play and to read the number aloud.

From session to session, the pattern of those first five minutes remains more or less the same, but each time the challenge is slightly increased. For instance, negative checkers are introduced . . .



(= 9,000)

. . . and Minicomputer configurations are presented . . .



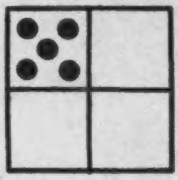
(= 2,150)

. . . that invite the student to mentally make more than two forward plays.

After an intense five minutes of this experience of decoding numbers on the Minicomputer, we change to a completely different activity such as Minicomputer Golf (see Section 2.4) or a detective story (see Section 3.2).

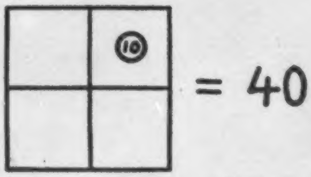
2.3 Puzzles

Isn't "5 x 8" a delightful shorthand for "8 + 8 + 8 + 8 + 8"? Putting five checkers on the 8-square of the Minicomputer gives a visual support by clearly picturing both of these ways of expressing the number.



In the early grades, multiplication is introduced on the Minicomputer and and the compact expression is quickly adopted by most of the students using CSMP. Progressively the times symbol becomes a familiar part of their mathematical vocabulary. But as we began the CSMP entry program for Grade 4 with two development classes, some of the slower students were not even at the level of our average second graders and had very limited knowledge of the mathematical vocabulary. Each time a term like "3 x 7" was used, they had to be reminded of its meaning.. It is superfluous to say that their conceptual understanding of multiplication was poor and that they had memorized hardly any multiplication facts.

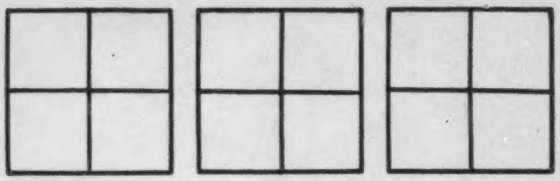
During the first year of math play therapy sessions, we did not focus our work on multiplication, because we felt that the students more urgently needed to experience success and enjoyment with numbers and that preparation work had to be done on the Minicomputer and with arrows. But this year, in the collective teaching, we were surprised to observe that students who still mix up such easy expressions as "2 x 8" and "2 + 8" immediately understood that one ⑩ -checker on the 4-square of the Minicomputer, for instance, represents "10 x 4 = 40".



The language of the weighted checkers seemed to speak directly to the students. So in the small groups, we started using the new checkers in a leisurely way and progressively developed the activities that we will now present.

The teacher displays three Minicomputer boards and the checkers:

② ③ ④ ⑤ ⑥ ⑦ ⑧ ⑨



The students do not have a hand-calculator.

Activity 1

The teacher puts a ③ -checker successively on the 1-square, 2-square, 4-square, and 8-square, each time asking the students which number is on the Minicomputer. (Answer: 3; 6; 12; 24) Then she proceeds to the tens' board and the hundreds' board, moving the checker over the squares and asking at each square the same question. (Answer: 30; 60; 120; 240; 300; 600; 1,200; 2,400) The exercise is repeated with another checker.

The binary code for the individual boards of the Minicomputer and the decimal code from board to board help the students find the successive answers. The jumps from "3 x 2 = 6" to "3 x 4 = 12" to "3 x 8 = 24" are easy. The passage from the ones' board to the tens' board to the hundreds' board is natural and emphasizes an interesting pattern. Repeated exposure (about every two weeks) to a similar activity with different checkers not only helps the students to memorize some multiplicative facts, but makes them increasingly aware that if they multiply one factor of a product by 2 or by 10, the product itself is multiplied by 2 or by 10, respectively. Later on this idea easily transfers to numbers other than 2 or 10.

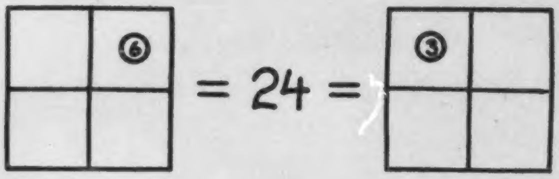
Activity 2

T: Can you put 24 on the Minicomputer using exactly one of these checkers?

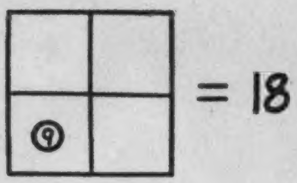
Some students whisper the correct answer to the teacher. If a student does not find a correct solution, she says:

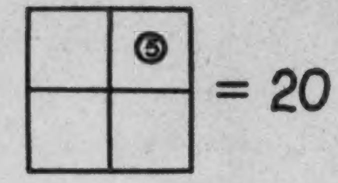
T: Greg has found a way of showing 24 with a ⑥ - checker (or with a ③ - checker). Do you know on which square Greg will put the ⑥ - checker (or the ③ - checker)?

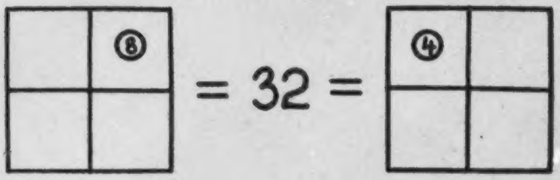
The teacher selects a student who had trouble earlier to solve this simpler problem.



She proceeds in a similar manner with other numbers, for instance, 18, 20 and 32.







Then she asks the same question about 26. The number 26 is most intriguing for the students. They generally make several tries before concluding that there is no way of putting 26 on the Minicomputer using exactly one of these checkers.

Decomposing a number into two of its factors is a challenging exercise for these students. Here again the Minicomputer provides the assistance they need in this endeavor, and the moveable checkers allow many tries that leave no trace of wrong answers.

Activity 3

The teacher writes a list of numbers on the board:

12; 15; 28; 36; 42; 48; 120; 150; 280; 1,200

T: From this list, try to find all the numbers that can be put on the Minicomputer using exactly one of these checkers.

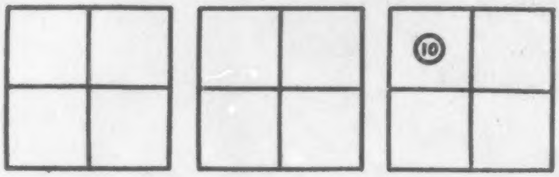
This is a simple variation of the preceding activity, but here the students can start their investigation with any number from the list. The added freedom is generally beneficial. Some students gain confidence by immediately putting an easy number like 12 on the Minicomputer or by rejecting another easy number like 15. Other students prefer to challenge themselves by working on the big numbers.

Activity 4

The teacher adds a negative checker and a ⑩-checker to the set of checkers she has displayed.

T: Can you put the number 92 on the Minicomputer using this ⑩-checker and exactly one of the other checkers?

If a student volunteers an answer, the teacher first asks the student to put the ⑩-checker on the boards. Suppose that the ⑩-checker is put on the 8-square.

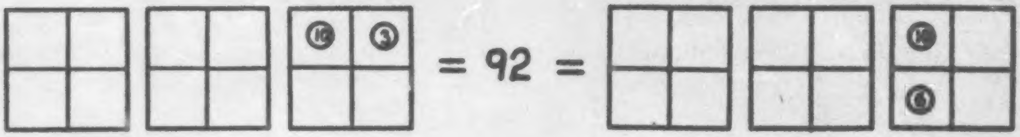


T: What number is this? (Answer: 80)

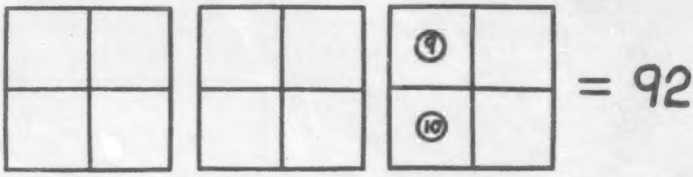
How much more do you need to get 92? (Answer: 12)

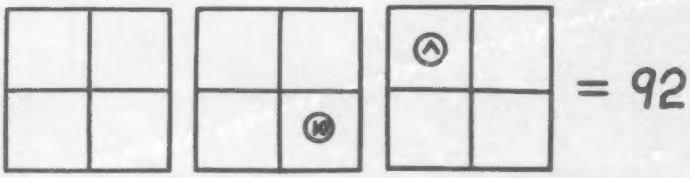
Who can put on 12 using exactly one of the other checkers?

Another student is asked to do it. There are two possibilities:



The teacher proceeds in a similar manner if the first student puts the (10)-checker on the 2-square or on the 10-square.



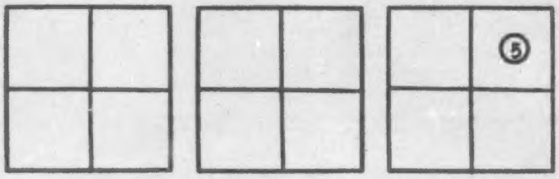


The activity continues with a similar question being asked for other numbers; for example, 76; 424; 380; 1,048. Later on, the teacher replaces the (10)-checker by a (20)-checker or even by a (50)-checker.

Activity 5

T: Can you put the number 52 on the Minicomputer using exactly one ③ - checker and one ⑤ - checker?

The students try different combinations of the two checkers. For instance, one student suggests putting the ⑤ - checker on the 4-square.



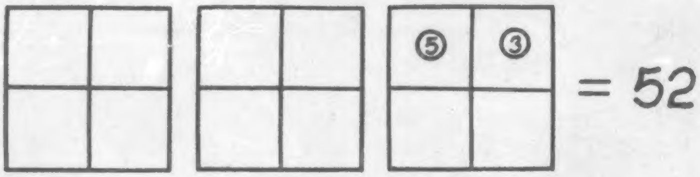
T: What number is this? (Answer: 20)

How much more do we need to get 52? (Answer: 32)

Can you put on 32 using exactly one ③ - checker?

S: No, it would be possible with a ④ - checker but not with a ③ - checker.

Other combinations are tried and the students eventually find the unique solution to this problem.



This activity is repeated two or three times with other numbers but the same pair of checkers. Later on, the teacher will present similar puzzles using other pairs of checkers, or even more than two checkers.

In Chapter 3, the reader will see weighted checkers being used in the clues for detective stories.

2.4 Minicomputer Golf

Perhaps the most important activity for helping the students increase their fluency in the language of the Minicomputer has been the Minicomputer Golf Game. There is a quite lengthy discussion of this game in Volume I, see Section 2.2 Minicomputer Golf, pp. 39-57.

Our initial experience with the game this year was important in terms of illustrating where the students were after the summer vacation. But first let us introduce the participants in that first game of the year. The four boys were Harry, Dave, Max, and Slim. It is unnecessary to say more about Harry, who was introduced in Chapter 1 (see pp. 37-46).

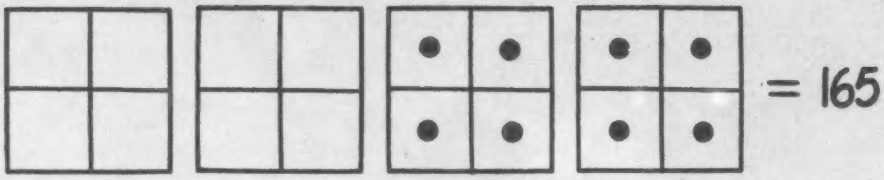
The irrepressible one of the group is Dave, an artist in body language and no slouch with the one-liner either. He is the sort of boy who brightens the day for those who are susceptible to his charm and can wrench a chuckle from even the most immune person in his vicinity. Having a positive view of himself, he can tolerate his mistakes with optimism and retain great curiosity and willingness to experiment in a new situation. As is perhaps a natural consequence of his sense of comedy, he is sometimes easily deflected from a serious pursuit of knowledge to the more quickly rewarded effort of spreading laughter. Nevertheless, his light approach to life can be a welcome counterbalance in a group consisting mostly of rather intense, unconfident children.

Max has the puzzling ability to appear an extrovert at one moment and an introvert at another. Sometimes he will be very involved with other students and eager to participate in the collective lesson; then suddenly he may retreat into himself. Part of the reason for his behavior is that he has a very shaky ego which seems to need continuous praise from outside in order to have the courage to keep exposing itself to risk. But there is a tenacity about him, too, and an obvious interest in learning which has grown as his confidence has developed.

Slim is probably the least confident of the group, having experienced much failure both at school and at home. His reading skills are very weak and this has affected all his other work severely. Sometimes he prefers to spend extra time on reading, so we leave the choice up to him. Consequently, he is below the ability level of the other boys in this group. He does have a good intuition about numbers, however, and always enjoys the small groups once he begins to participate. The scowl of concentration and self-disgust that is so often expressed on his face can be replaced with the relaxation of a bright smile of satisfaction in a successful group experience.

But now to the game! The boys have already received hand-calculators. The teams are chosen: Dave and Max for Team A (red arrows); Slim and Harry for Team B (blue arrows).

The teacher arranges checkers on the Minicomputer two at a time, in no particular sequence, and various boys announce the number each time it is altered. Ordinarily only three boards are used, but this time the teacher inadvertently fails to remove the fourth board, which had been used for a previous exercise. This is the beginning configuration.



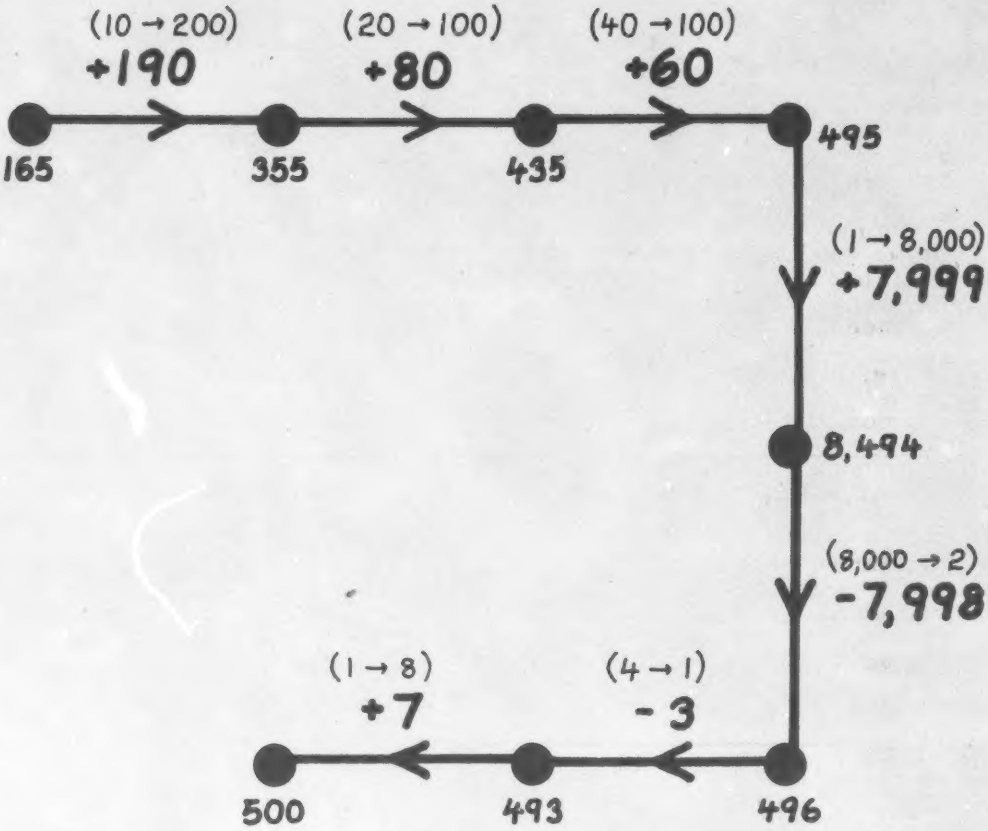
The group chooses 500 for the goal. Three plays follow in quick succession. First Dave moves a checker from the 10-square to the 200-square and says, "+190." Then Slim moves a checker from the 20-square to the 100-square and announces, "+80." In his turn, Max goes from the 40-square to the 100-square, saying, "+60." Thus they arrive at 495, demonstrating their ability to cover a large distance quickly and to accurately calculate each step.

Knowing that no winning move is possible and seeing the danger of setting up the board so his opponents can win, Harry faces a dilemma. He solves it by taking advantage of the extra board to make the largest possible move. He calmly walks to the Minicomputer and moves the checker from the 1-square to the 8,000-square. When asked about the increase, he promptly replies, "+7,999."

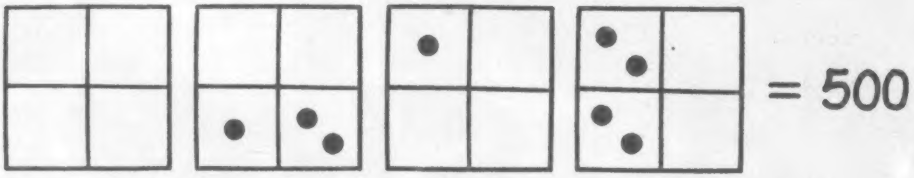
Not to be outdone, Dave prances up, moves the checker from the 8,000-square to the 2-square and chortles, "- 7,998!"

Probably confused by the appearance of such large numbers, Slim chooses the correct checker, but moves it in the wrong direction. Beginning at 496, Slim moves a checker from the 4-square to the 1-square (instead of to the 8-square). From 493, Max easily scores the winning move by taking a checker from the 1-square to the 8-square. Alas! Poor Harry and his team are defeated in spite of his shrewd move.

The complete game is described by this red-blue road.


(10 → 200)
+190
(20 → 100)
+80
(40 → 100)
+60
165
355
435
495
(1 → 8,000)
+7,999
8,494
(8,000 → 2)
-7,998
(1 → 8)
+7
(4 → 1)
-3
500
493
496

The final boards have this configuration:



As we look at this game, several things stand out much to our relief and encouragement. First it is clear that after the summer break of three months the students do not show a decline in their ability to play the game with understanding. To the contrary, they play with more confidence, make better estimations of the distance remaining to be covered, and calculate their individual moves more accurately than they did at the end of the previous school year. Moreover, the moves show that they are looking ahead in the game and trying to formulate a winning strategy. Even Slim shows a clear understanding of the strategy necessary to win when he makes his fatal move.

When we compare this game with one played eight months earlier by a group of very similar make-up[*], the mental development shown is even more striking. Not only has the number of moves dropped dramatically from 16 to 7, but the character of the moves has changed significantly. (The next illustration describes completely that 16 step game.)

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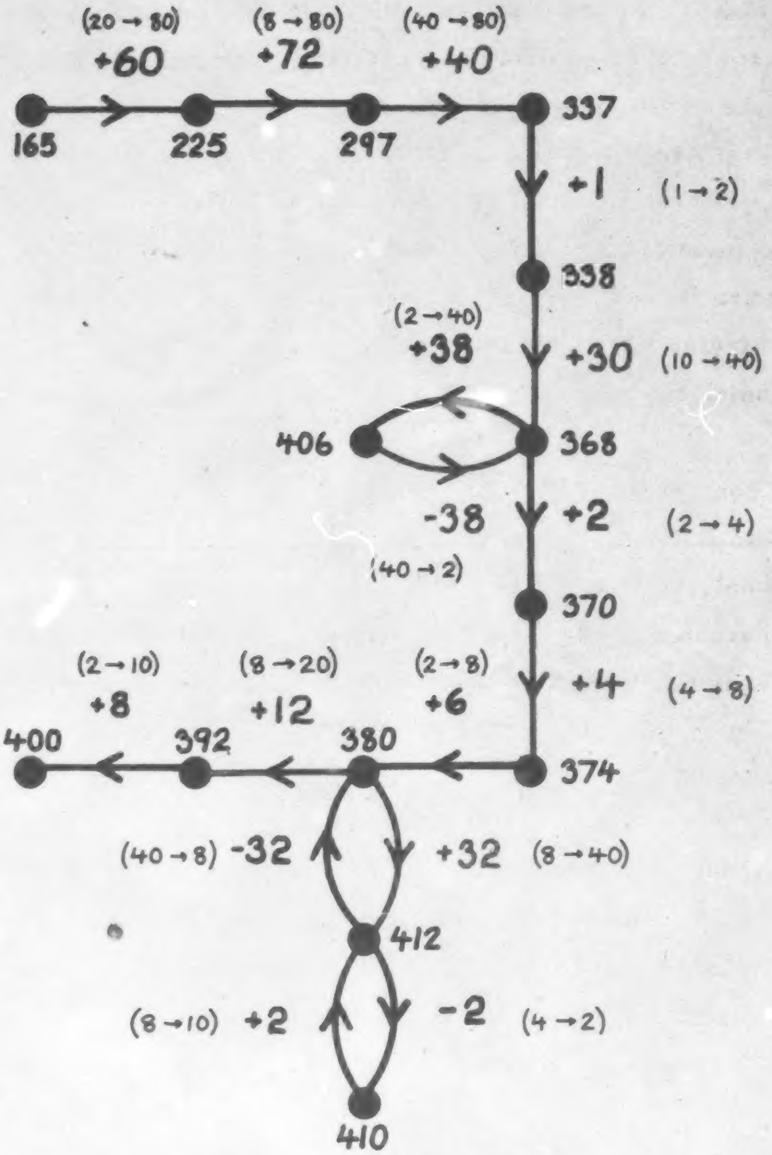
[*] Two students were in both groups and the two other members of each group were similar in ability and intuition. (See Volume I, p. 42.)


(20 → 80)
+60
(8 → 80)
+72
(40 → 80)
+40
165
225
297
337
+1
(1 → 2)
338
(2 → 40)
+38
+30
(10 → 40)
406
368
-38
(40 → 2)
+2
(2 → 4)
370
(2 → 10)
+8
(8 → 20)
+12
(2 → 8)
+6
+4
(4 → 8)
400
392
380
374
(40 → 8) -32
+32 (8 → 40)
412
(8 → 10) +2
-2 (4 → 2)
410

Here we can see the importance of long-term evaluation in the case of children who have learning difficulties. If we had based our idea of the students' improvement on short-term checks we would have become extremely discouraged, and needlessly so. For after some months their solid progress was easily recognizable and stimulated a new surge of ideas and determination to keep working.

During the fall, our students continued playing the Golf game every two or three weeks. Spontaneously, they began to estimate the distance to the goal. They did this with increasing frequency and accuracy in later sessions. Negative checkers were again included in the game and by the end of November, the children were ready for a new challenge. Happily, the introduction of weighted checkers in the collective class lessons coincided with the group's readiness to move forward. In the next section, we will present a new variation of Minicomputer Golf made possible by the weighted checkers.

2.5 A Variation of Minicomputer Golf

This natural variation of the Golf Game is presented after the students have had experience with the weighted checkers.[*] A simple substitution of two or three weighted checkers for two or three regular checkers greatly broadens the scope of the game, which now involves multiplication as well as addition and subtraction.

The initial response of a group to a new activity is always interesting. To give continuity to the book, we present a group of four boys who have appeared earlier as they first encounter the game in its altered form. Dave, Slim and Max were introduced at the beginning of the last section. Jack participated in sessions discussed in Section 1.3, but he was not described at that time.

An appealing, good-natured extrovert, Jack surely matches Dave in his ability to talk body language. Not too surprisingly, he is almost completely "turned off" by pencil and paper activities for learning. However, when he can become actively and verbally involved with ideas, he shows great interest and ability, excellent intuition, and quick perception. He is greatly disappointed when it is not his turn to come to math play therapy sessions.

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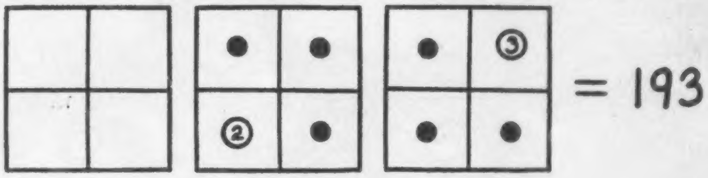
[*] See Section 2.3 Puzzles, pp. 61-68.

Let us now study the game. In a usual warm-up exercise, the boys announce the progressive series of numbers that appear on the Minicomputer as the teacher slowly places regular and weighted checkers on the boards. Because the task is somewhat more complex with weighted checkers, she gives them plenty of time to compute mentally. Finally the following configuration is on the boards and the students display the number on their hand-calculators.



After they select a goal of 600 and choose partners (Dave and Slim vs. Max and Jack), the game begins. Dave opens with a move from 10 to 200, announcing, "+ 190." The new number on the Minicomputer is 383. Max selects the ② - checker and moves it to the 80-square. He is not sure of the increase, offering tentatively, "+ 160?"

T (returning the ② - checker to its original position): What number does the ② - checker show here?

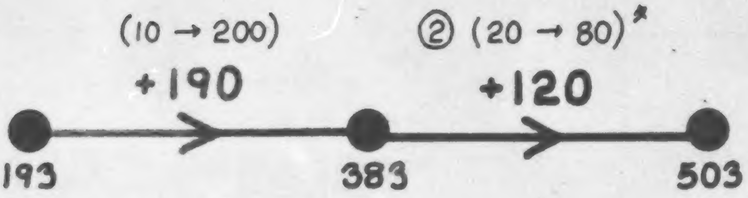
Max: 40, because 2 x 20 = 40.

T (moving the ② - checker back to the 80-square): And what number does it show here?

Max: 160, because 2 × 80 = 160.

T: So how much bigger is the new number?

Max: 120 bigger.



Now the players are at 503. Slim sees the need to be cautious but still wants to try moving a new checker, so he moves the ③ - checker from the 4-square to the 8-square. He, too, is unsure about the increase but thinks the resulting number might be 16 bigger. Once again the teacher simply separates the two positions on the board.

---

[*] In arrow picture records of Minicomputer Golf games with weighted checkers, "②(20 ⟶ 80)" indicates that a ② - checker was moved from the 20-square to the 80-square and "(10 ⟶ 200)" indicates a regular positive checker was moved from the 10-square to the 200-square.

T (returning the ③ - checker to the 4-square): What number does the ③ - checker show here?

Slim: 12, because 3 × 4 = 12.

T (moving the ③ - checker back to the 8-square): And what number does it show here?

Slim: 24, because 3 × 8 = 24.

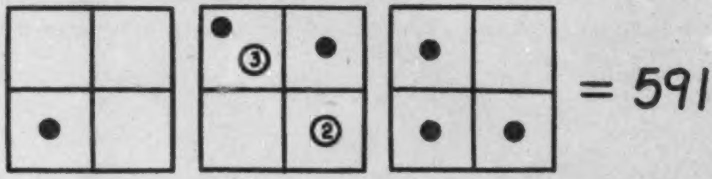
T: What is the increase?

Slim: + 12.

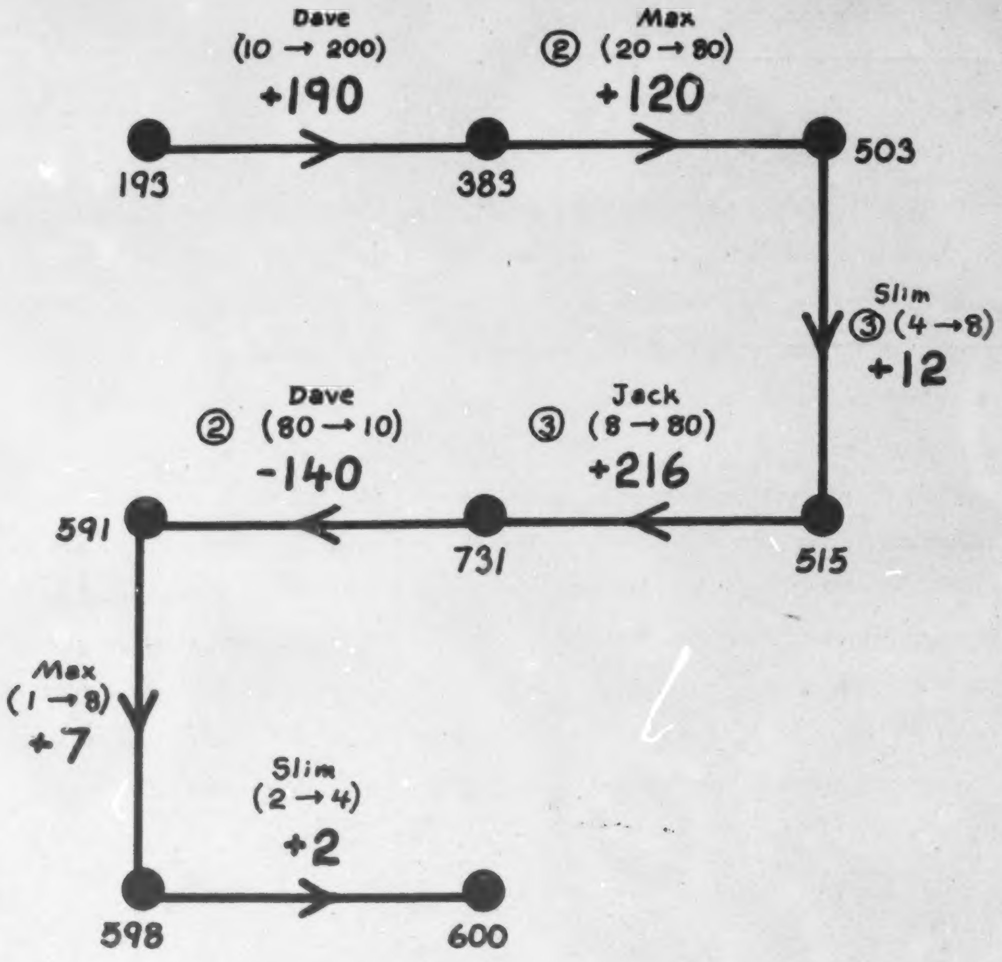
Slim's move takes them to 515 (503 + 12 = 515). It is Jack's turn and he decides to throw caution to the wind; he wants to see what these weighted checkers really can do. He quickly finds out by moving the ③ - checker from the 8-square to the 80-square making the number 216 bigger. His move takes them 131 over the goal!

In his determination to get the goal back in range, Dave forgets the possibility of getting too close and giving the opportunity for victory to his opponent. By moving the ② - checker from the 80-square to the 10-square, he decreases the number by 140 and brings them to 591, an easy stepping stone from the goal.

At this point, the configuration on the Minicomputer is:



The presence of the (2) - checker on the 10-square is obviously disturbing to Max. Instead of moving a regular checker from the 1-square to the 10-square, he moves it to the 8-square, setting up an easy victory for Slim.


Dave
(10 → 200)
+190
Max
② (20 → 80)
+120
193
383
503
Slim
③ (4 → 8)
+12
Dave
② (80 → 10)
-140
Jack
③ (8 → 80)
+216
591
731
515
Max
(1 → 8)
+7
Slim
(2 → 4)
+2
598
600

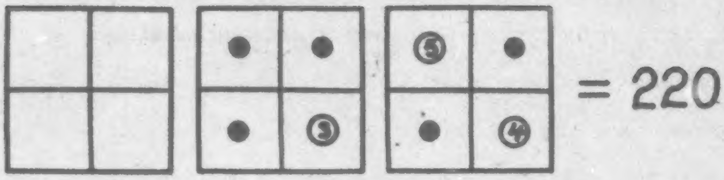
Even with the added complexity of weighted checkers, the boys build a road from 193 to 600 in only seven steps. Instead of being intimidated, they quickly plunge into using the new checkers, apparently wanting to see what new information they can glean. They show real spunk and tenacity, even refusing to resort to a move with a more familiar regular checker when they greatly overshoot the goal. Only when the situation demands (i. e., they can win) do they return to regular checkers.

One week later we have a session with Harry, Max, Jack, and Brian. Although he has not been mentioned before, Brian sometimes alternates with other members of this group.

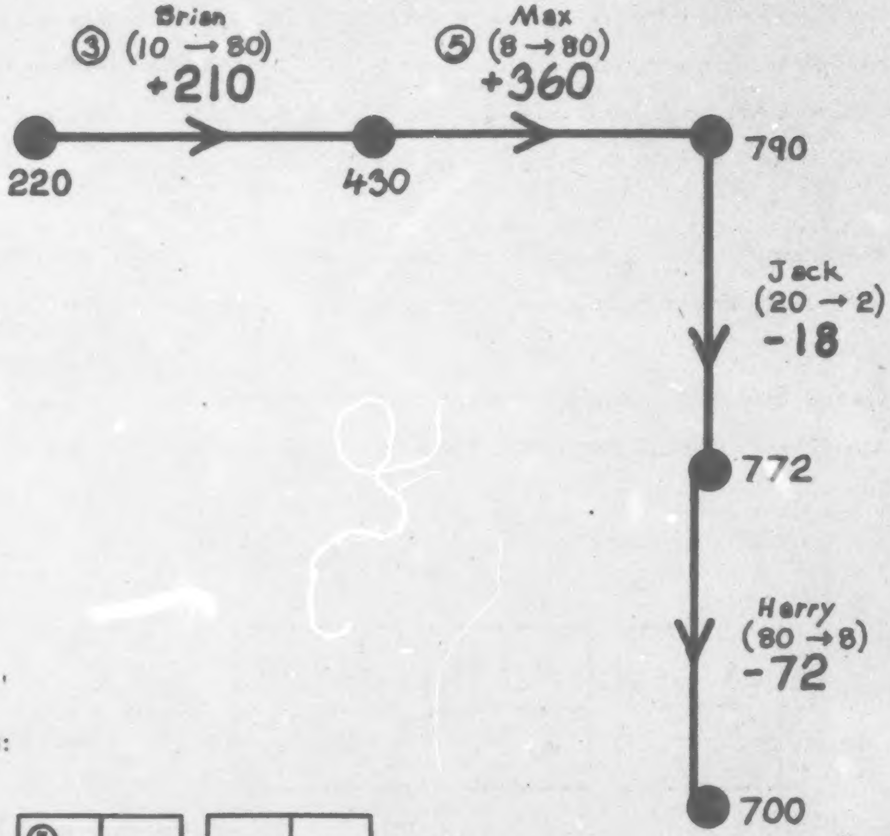
A slow-moving, quiet-spoken, unflappable child, Brian can rarely be stimulated into an enthusiastic response to anything. He is very far behind in both reading and mathematics skills. It appears he is unwilling to settle himself quickly into a task and concentrate until it is finished, much preferring to manipulate an adult or fellow student into helping him. Even in the small groups he often plays a passive role, but here he is more likely to be enticed into participating than in the collective lessons. On this particular day, for instance, he sets the ball rolling very nicely.

The game proceeds very much like the first one. The partners are chosen (Brian and Jack vs. Max and Harry), the checkers are arranged on the boards, the number is entered on the hand-calculators, and the goal is picked. The game begins with an opening move by Brian. The starting and ending configurations and the red-blue arrow road pictured below describe the game completely.

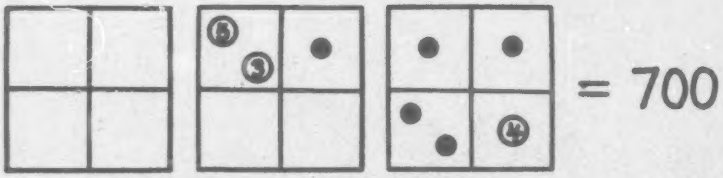
Starting boards:



Goal: 700


Brian
3 (10 → 80)
+210
Max
5 (8 → 80)
+360
220
430
790
Jack
(20 → 2)
-18
772
Harry
(80 → 8)
-72
700

Final boards:

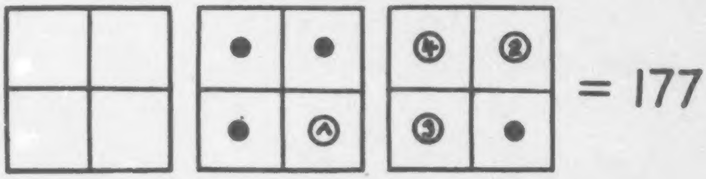


Here again the boys exhibit the same enthusiastic and self-confident response to a still unfamiliar situation. In fact, this was the first exposure for Brian and Harry. Nonetheless, they show their preference for the challenge of trying the new weighted checkers.

Brian and Max quickly bring the group close to the goal with the help of the new checkers. Jack resorts to a more cautious move in order to stay far enough away from the goal to give his team at least a fighting chance to win. But Harry is not to be denied a victory. He studies the configuration until he spots a winning move among the ③, ⑤, and regular checkers that are bunched together on the 80-square. Once again the students master a difficult situation with determination and growing skill.

We pass now to a game played just after the Christmas break in a session with Jack, Dave, Max, and Harry.

Harry and Jack make up the red team; Dave and Max the blue team. The starting configuration is for the number 177 . . .



. . . and the goal is 1,000.

Harry opens by moving a checker from the 1-square to the 800-square, and this move brings them to 976.

Harry: We need just 24 more.

Jack: They can do it in one move.

The other boys are doubtful, but Jack is convinced he is right even though he's not sure how to do it.

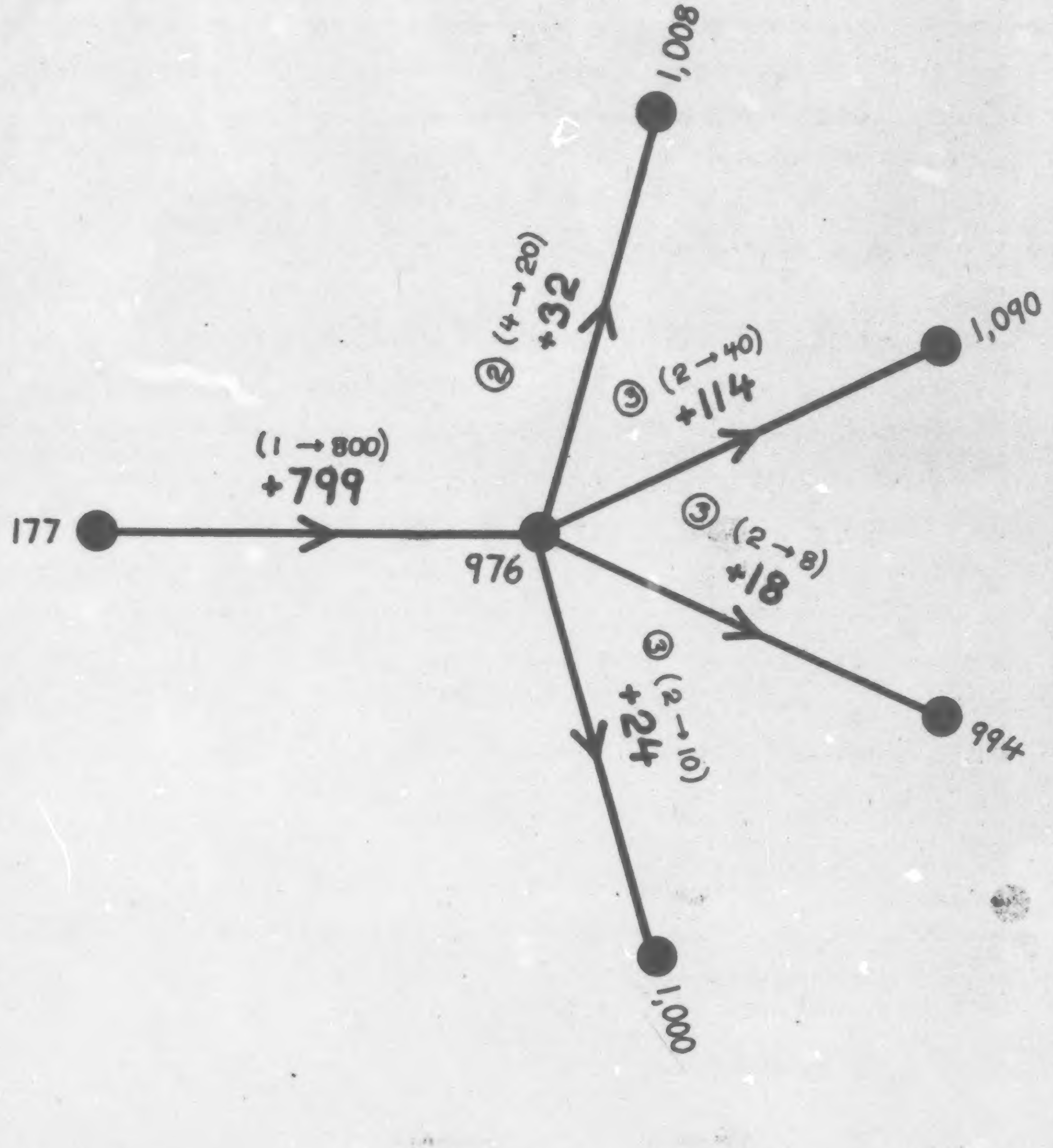
T: It would be interesting to see if Jack is right. Let's change the rules of the game for the moment. You may each take turns, beginning with Dave, until either someone reaches the goal in one move or we are convinced it isn't possible.

Dave chooses the ② - checker and moves it from the 4-square to the 20-square. He discovers this increases the number by 32, and brings them to 1,008.

Jack believes the ③ - checker is the one to use. He is correct, but unfortunately he greatly overshoots the goal with his move from the 2-square to the 40-square, an increase of 114. He is soundly scolded by Harry, who now sees the correct move and had expected Jack to make it.

The ③ - checker also seems the best choice to Max but, alas, he is too cautious. He moves it from the 2-square to the 8-square and finds himself six short of the goal.

Now, Harry has his chance and confirms the claim by moving the ③ -checker from the 2-square to the 10-square, winning the game.


1,008
② (4 → 20)
+32
③ (2 → 40)
+114
1,090
(1 → 800)
+799
177
976
③ (2 → 8)
+18
994
③ (2 → 10)
+24
1,000

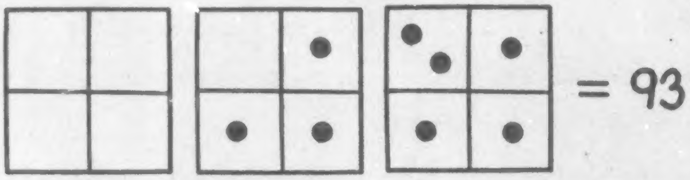
Jack's spontaneous remark leads the group to discover the shortest possible road from 177 to 1,000. Their reactions also seem to indicate that these children are ready for a new level of exploration which would help them to focus on finding shortest roads under diverse circumstances. A variation of the game which encourages the players to compete with themselves more than with each other will be tried next year.

2.6 Barry and Ralph

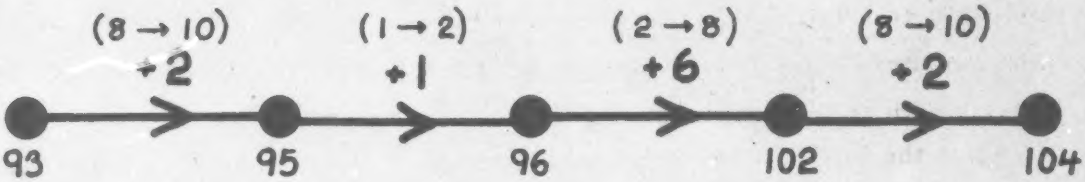
The preceeding games in this chapter were played by students who have made considerable progress during the period of time we have worked with them. We wish now to again observe Barry and Ralph, who appeared earlier in this book, to study briefly two of their much less sophisticated games at Minicomputer Golf. These games are included as a reminder that even when working with children who have severe limitations in mathematics, a teacher can adapt the Golf game to offer students with only a rudimentary understanding of numbers a variety of challenging opportunities.

Barry and Ralph made some progress when they separately worked in a group of four, but it became increasingly clear that they were unable to take responsibility for the work when the three other members were at a much higher level of understanding. However, when they were paired with each other and given game situations at their own level, they began to show a marked improvement in ability.

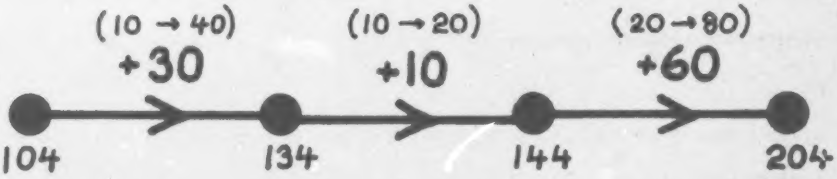
The following two games were played during their second session together and one week before they used the hand-calculator activities described in Chapter 1. The starting configuration of the first game is for the number 93 and the goal is 200. Although the starting number is a relatively small one, the teacher was careful to choose a configuration using eight checkers to give more flexibility.



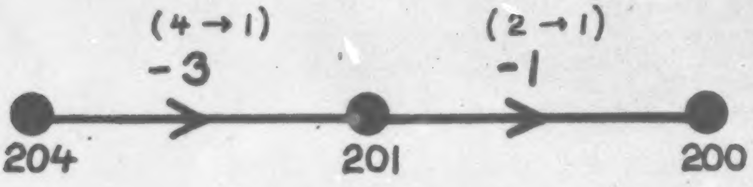
Barry's moves are represented by blue arrows and Ralph's by red arrows. Both boys are very hesitant in the beginning, as is illustrated by their four opening moves that together increase the number by only 11.


(8 → 10)
+2
(1 → 2)
+1
(2 → 8)
+6
(8 → 10)
+2
93
95
96
102
104

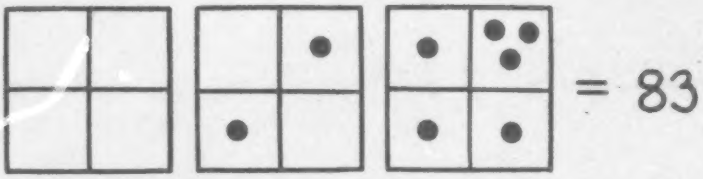
With their next three moves they become more venturesome, especially Barry.


(10 → 40)
+30
(10 → 20)
+10
(20 → 80)
+60
104
134
144
204

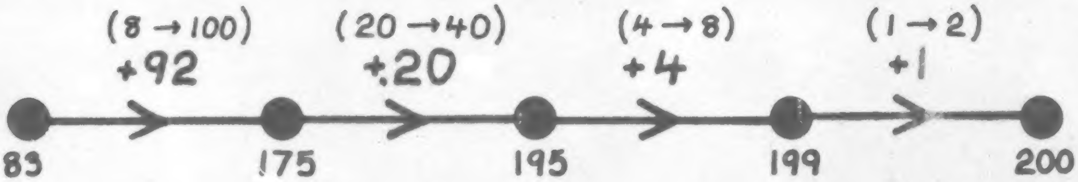
Ralph has an opportunity to win but he does not see it, nor is he able to select a good defensive move; so Barry claims the victory.



The goal remains the same for the second game, but the starting number is changed to 83, with the following configuration.



Ralph shows a burst of confidence by moving a checker from the 8-square to the 100-square on his first turn. Whether by accident or design we are not sure, but Barry makes an excellent choice when he moves a checker from the 20-square to the 40-square. With the resulting configuration of checkers, it is not easy for Ralph to make a defensive move from 195, and a winning move is impossible. Because Ralph is unable to protect himself, Barry wins again.


(8 → 100)
+92
(20 → 40)
+20
(4 → 8)
+4
(1 → 2)
+1
83
175
195
199
200

The outstanding improvement in the second game shows what can happen when our most limited children are given a suitable set of circumstances.

3

DETECTIVE STORIES

3.1 Introduction

It has been stated many times that the ability to count is an essential part of the foundation which must be laid in order to properly build an understanding of numerical attributes and processes. We believe this is true because the counting experience is so intimately tied to both our inner, private world and our outer, public world. The sequential characteristic and the repetition of counting can be experienced inwardly as they are related to breathing, to the beating of the heart, to walking or running gaits, and so on. The progression of days, weeks, seasons, and years are only a few of the outer expressions of order we can recognize as being related to regular sequences of numbers.

In the realm of music, we also discover the importance of repetition and sequence -- the repetition of basic rhythms and the scale sequence upon which tonal patterns are built. The child who has been in a musically rich environment has internalized these two different aspects of music and consequently exhibits a greater ability to respond to and understand music than do children who have not had such opportunities. In like manner, the child who has internalized an understanding of the counting process through early, varied, and frequent experiences in the home is more likely ready to deal with numbers in more advanced ways. Even in first grade it is easy to locate the children who have not had such an environment. Sadly, by first grade it is already difficult, if not impossible, to provide such experiences for they arise in free-play situations which are not easy to arrange when most children are ready for more structured work. Thus begins a vicious circle. Students who are not ready to deal with structured numerical activities slip farther and farther behind and this may do irreparable damage to their self-esteem and personalities.

When CSMP began developing a fourth grade entry program, we found that most of the students who had difficulty with the numerical strand also had trouble counting forward and backward by fives, tens, twos, threes, etc. Clearly, they had missed some vital early steps. Our problem was how to repattern the missing steps. Luckily, we had incorporated the hand-calculator into the program and as we began to experiment with it in various ways, we saw that it had potential for helping to improve this situation.

Because the calculators we use have function keys which are permanently in a constant mode, the students have the ability to generate patterned sequences of numbers either forward or backward, simply by pressing $\boxed{+}$ $\boxed{n}$ $\boxed{=}$ $\boxed{=}$ . . . or $\boxed{-}$ $\boxed{n}$ $\boxed{=}$ $\boxed{=}$ . . . . Thus, in exploratory activities with hand-calculators, they are provided with exactly the sort of sequential experience they lack and the enticing quality of the calculators focuses their attention on the sequences. Taking advantage of this dimension of the hand-calculator, we designed arrow clues in the detective story format which would give the students many opportunities to reproduce and observe different sequences of numbers. We began with a simple situation using one arrow so that it would be easy for the students to predict that there is a pattern and to identify it (see Section 3. 2).

Thanks to the flexibility of the arrow language, we were next able to design a more subtle and challenging clue using two arrows. The students were quickly intrigued by this situation because the pattern was not obvious and hence a challenge to find. Detective stories using such clues are discussed in Section 3. 3.

3.2 The First Ten Minutes

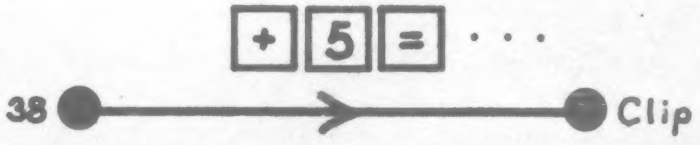
This section consists of three short detective stories as prototypes of those that use clues developed for first level exploration of patterned sequences of numbers. In each of these examples, the responses represent a compilation of excerpts from several different groups of students.

From the beginning, we felt it was necessary to offer children experiences with patterns that they had somehow missed in their early years. We found that an effective way to do this was to devise activities which were short and intense, but which also retained a spirit of playful inquiry. We never repeated an episode but regularly offered variations which gave the children a fresh exposure to the same idea at a slightly different level. This proved to be more beneficial than to spend a longer period of time trying to exhaust every possibility in a given set of circumstances.

During the first semester of this school year, we devoted about ten minutes at the beginning of every other session to this intensive work on patterns. We quickly recognized a steady improvement in the students' awareness and eventually found that all the students were able to recognize specific patterns. Some students became experts at identifying patterns quickly.

Activity 1

Clip is a secret number. The teacher puts the first clue on the board.



T: What numbers could Clip be?

The students play freely with their hand-calculators and observe the numbers that are generated when they press $\boxed{3}$ $\boxed{8}$ $\boxed{+}$ $\boxed{5}$ $\boxed{=}$ $\boxed{=}$ .... They give examples of numbers that Clip could be; small numbers and large numbers. They are encouraged to comment about these numbers.

S: They're all even.

S: No, sometimes they end in 3 which is odd.

S: I know! 8, 3, 8, 3, 8, 3, and so on.

S: They all end in 3 or in 8.

When the pattern is clearly recognized by all the students, the teacher focuses their attention on some more specific question. For instance,

T: Display 38 and then hide the display with your hand. Press $\boxed{+}$ $\boxed{5}$ $\boxed{=}$ $\boxed{=}$ $\boxed{=}$ $\boxed{=}$. Which number should be on the display?

S: 58.

T: Look and see if it's correct. Then hide the display again.

The teacher proceeds in a similar manner until they reach 83.

T: Hide the display again. Press $\boxed{=}$ as many times as necessary to get 113 on the display. Check the display to see if you were correct. Do you know how many times you pressed $\boxed{=}$?

S: Six times.

T: Now you have 113 on the display. Can you estimate how many times you must press $\boxed{=}$ to get 213?

The teacher writes some of their estimates on the board.

T: Check your estimate by trying it on the calculator.

After a little while, a student discovers that $\boxed{=}$ must be pressed 20 times.

T: Can you explain why you need to press $\boxed{=}$ 20 times to go from 113 to 213? (Answer: Because $20 \times 5 = 100$)
If you keep pressing $\boxed{=}$, which is the largest number smaller than 1,000 that will appear on the display?
Try to answer without pressing $\boxed{=}$ any more times.

Students who cannot find the answer mentally are allowed to use the hand-calculator. (Answer: 998)

The teacher then gives the second clue.

T: Here is a new hint about Clip. If you keep pressing $\boxed{=}$, Clip is the smallest number larger than 2,000 that will appear on the display. Who is Clip? Try to find it without actually pressing $\boxed{=}$.

With or without the help of the calculator, the students discover that Clip is 2,003.

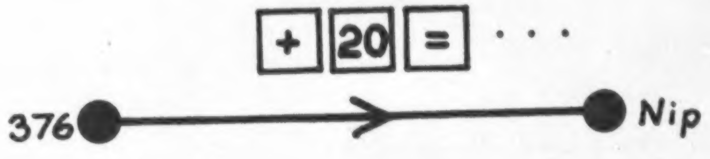
This short warm-up activity has a natural attraction for the students. They are fascinated both by the mysterious code, $\boxed{+}$ $\boxed{5}$ $\boxed{=}$ . . . , and by the everchanging pattern of numbers the hand-calculator flashes before them when they use this code. They are determined to discover that pattern. Some find it quickly and others need more time, but all can eventually find it.

The students also enjoy playing "hide and seek" with the display in the second step of the activity. Under these circumstances, they find it easy to follow instructions closely. Their concentration improves their mental calculations. However, the teacher always allows them to check with the hand-calculator if their mental calculation is incorrect.

The second clue for finding Clip might appear too much of a challenge for these students, but because it only involves ordering and not computation, it is within their range of abilities. Once they know the pattern: 3, 8, 3, 8, . . . , some are able to find Clip mentally. A few may again use their hand-calculators.

Activity 2

The teacher puts the first clue on the board.



The students start exploring the clue with their hand-calculators. They concentrate on the numbers that appear on the display and one by one they discover the pattern.

S: Each number ends in 6.

S: And the second digit is 7 or 9 or 1 or 3 or 5.

T: What are some numbers Nip could be?

The students offer many examples of such numbers; they especially enjoy offering very large numbers. After some moments, the teacher begins the "hide and seek" game.

T: Display 376 and hide the display.
Press $\boxed{+}$ $\boxed{20}$ $\boxed{=}$ $\boxed{=}$ $\boxed{=}$ $\boxed{=}$ $\boxed{=}$. What number should be on the display?

S: 476.

T: Look and see if it's correct. Then hide the display again.

The teacher proceeds in a similar manner until they reach 596.

T: Hide the display again. Press $\boxed{=}$ as many times as necessary to get 796. Check your answer. Do you know how many times you pressed $\boxed{=}$?

S: Ten times.

T: Hide the display again. Press $\boxed{=}$ as many times as necessary to get the first number with four digits.

After giving the students a few minutes to think about the problem, the teacher asks:

T: Which number should be on your display now?

S: 1,016.

Wishing to give them a break from deep concentration, the teacher selects a more open question.

T: What are some numbers larger than 2,000 that Nip could be? Try to find them without using your hand-calculator.

As the students respond, the teacher writes some of their answers on the board.

2,096; 4,526; 10,376; 2,000,446

S: 4,526 is wrong!

T: How do you know?

S: The second digit is 2. It could be 4,536 or 4,516.

A similar comment is made about 2,000,446, so it is replaced by 2,000,456.

T: Here is a new hint about Nip. If you start with 376 and press $\boxed{+}$ $\boxed{20}$ $\boxed{=}$ $\boxed{=}$ and so on, Nip is the largest number smaller than 5,000 that will appear on the display. Who is Nip? Try to find it without actually using the hand-calculator.

With or without the help of the hand-calculator, the students discover that Nip is 4, 996.

Comments:

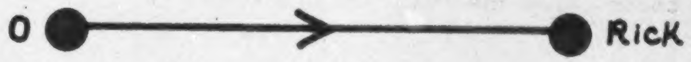
To avoid repetition we have omitted many other versions of the warm-up Activity 1 in which patterns are explored. Relations such as $\boxed{+}\ \boxed{10}\ \boxed{=}$ . . . , $\boxed{-}\ \boxed{10}\ \boxed{=}$ . . . , and $\boxed{-}\ \boxed{5}\ \boxed{=}$ . . . generate easily discernable patterns and were a natural lead in to Activity 2. With $\boxed{+}\ \boxed{20}\ \boxed{=}$ . . . , the pattern is more difficult to discover since it involves differently two digits. However, the students were easily able to observe the pattern of odd digits in the tens' place and 6 consistently in the ones' place.

When the teacher asked for numbers larger than 2, 000 that Nip could be, students spontaneously introduced very large numbers. They were pleased and proud to be able to project the pattern to such distances. With this preparation, Nip was quickly discovered once the second clue was given.

Activity 3

The teacher puts the first clue on the board.

$\boxed{+}\ \boxed{3}\ \boxed{=}$ . . .



The teacher allows time for free exploration of this clue and accepts the spontaneous comments of the students about the numbers that Rick could be.

S: They are all multiples of 3.

S: They end in different digits.

T: Do you think all the different digits or only some of them can be the last digit?
Press $\boxed{=}$ slowly and each time carefully observe the last digit.

S: I think all of them can be the last digit.

T: Continue pressing $\boxed{=}$ slowly and try to find a pattern.

If there is no response, she asks questions that direct their attention to the pattern.

T: What are some numbers ending in 7 that Rick could be?

The students suggest small and large numbers.

T: Which is the smallest number ending in 7 that Rick could be?

S: 27.

T: The next one?

S: 57.

T: The next ones?

As the students give numbers, the teacher writes them on the board.

27, 57, 87, 117, 147, 177, 207, 237, . . .

T: What do you notice?

S: Each time the next number is 30 more.

The teacher proceeds in a similar manner for the numbers ending in 2 that Rick could be.

12, 42, 72, 102, 132, 162, 192, . . .

S: We are counting by 30 again.

The teacher focuses the students' attention on the numbers ending in 0 that Rick could be.

30, 60, 90, 120, 150, 180, 210, . . .

S: They are multiples of 30.

T: Without using your hand-calculator, can you find some numbers larger than 1,000 that Rick could be?

S: 3,000.

S: 3,009.

Ralph: 1,002.

T: How did you get 1,002?

Ralph: Well, you can get 9 and you can get 900, so you can get 999. And 3 more is 1,002.

T: Can you suggest some numbers in the close neighborhood of 2,000 that Rick could be?

S: 1,992.

T: How did you get it?

S: 1,002 + 990 = 1,992.

T: Which is the closest neighbor of 2,000 that Rick could be?

S: 1,992 . . . 1,995 . . . 1,998 . . . 2,001. It's 2,001!

T: Rick is this number.

Comments:

In Activity 3 a new step is taken. Here the pattern is not so obvious and the teacher does not expect the students to describe it completely. Instead she gives more time for exploration and encourages them to play around with the clue by asking a variety of questions. These inquiries help the students to recognize certain aspects of the pattern and to discover other patterns formed by sub-sequences.

In the descriptions of these short activities, we do not attempt to label the students' responses, since these are composite descriptions of responses of several groups. However, in Activity 3 we do identify a short dialogue that took place with Ralph, because it is an illustration of a phenomenon that occurs many times with children at this level. Something about activities involving patterns seems to release the intuition of these children and they give answers that appear to be far above their normal level of understanding. Section 3.3 includes other examples of such occurrences.

3.3 Problems with Two Arrows

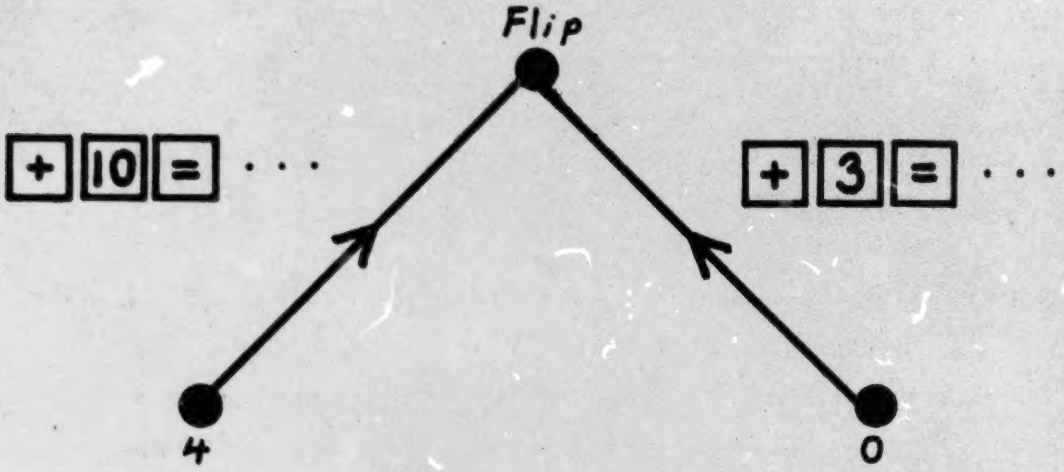
During the second semester we concentrated on some less obvious patterns in sequences of numbers referred to in this chapter's introduction. This section presents a series of three detective stories that were solved by one small group, largely the same each time, during three separate sessions. The students in this group were introduced in Chapters 1 and 2, where we featured their work in other areas.

Here the accounts of this small group are not verbatim records but rather an extraction of the important responses given by the students to the teacher's questions and remarks. Because we are interested in highlighting the individual improvement exhibited by the children, we have included some comments of a personal nature. The students were exposed to the simpler material described in Section 3.2 before they were introduced to these stories.

Problem 1: Who is Flip?

Harry, Max, and Dave are present.

The teacher draws this picture on the board and remains silent.



The three boys begin working with their hand-calculators. After several minutes the teacher asks:

T:	Which numbers could Flip be?
Harry:	84.
Dave:	144.

The teacher does not comment on these answers but writes them on the board.

T:	Which is the smallest number that Flip could be?
Harry:	24.
Dave:	34.
Harry:	It's 24; 34 is not a multiple of 3.

T: What are some other numbers that Flip could be?

Max: 54 and 114.

The teacher continues to write the students' suggestions for the numbers Flip could be on the board.

Flip could be : 24 ; 54 ; 84 ; 114 ; 144 ;

T: What do you notice about the numbers that Flip could be?

Dave: They all end in 4.

T: Do you notice anything else?

There is no answer.

T: Can you find some more numbers that Flip could be?

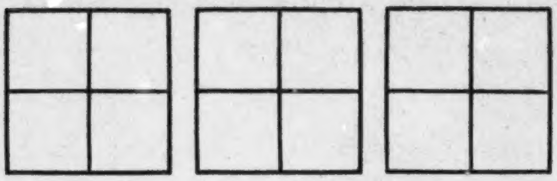
As the students suggest numbers, the teacher writes them on the board.

Flip could be : 24 ; 54 ; 84 ; 114 ; 144 ;
174 ; 204 ; 234 ; . . .

Harry: Oh, I see! We are counting by 30.

The teacher displays three Minicomputer boards and the weighted checkers:

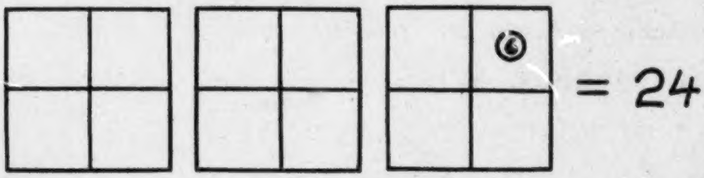
② ③ ④ ⑤ ⑥ ⑦ ⑧ ⑨



T: Here is another clue.
Flip can be put on the Minicomputer using exactly one of these checkers.

Dave: Flip could be 24.

He puts the ⑥ - checker on the 4-square.



T: Could Flip be 54?

The teacher encourages the students to try to find a way to put 54 on the Minicomputer with exactly one of the checkers. They successively put on 48 with a ⑥ -checker on the 8-square; 56 with a ⑦ -checker on the 8-square; 64 with a ⑧ -checker on the 8-square; 72 with a ⑨ -checker on the 8-square.

Harry: There's no way to put on 54.

T: What about the other numbers? Could Flip be 84; 114; 144; 174; and so on?

The students make attempts at putting on numbers with checkers on the tens' board. For instance, Dave shows 160 by putting a ④ - checker on the 40-square. Harry shows 280 by putting a ⑦ - checker on the 40-square.

Dave: With the checker on the tens' board all the numbers end in 0. We can't get any of our numbers that way. So Flip is 24.

Comments:

The teacher is careful to select a simple, clear, visual clue which she draws on a board directly in front of the group. By remaining silent, she allows the picture to do the communicating and gives the students an opportunity to organize their own thinking. She simply waits patiently to see what happens, always alert to the facial expressions and comments of the boys. As the transcript indicates, the students do not need preliminary help when they understand a situation. In fact, they first discover not the smallest numbers, but some of the larger ones, showing a clear grasp of the problem.

In the same way, when an error is made (for example, 34), she does not hasten to correct it, thus giving the students the valuable experience of correcting their own mistakes. A simple question leads them to find the missing numbers between 24 and 144.

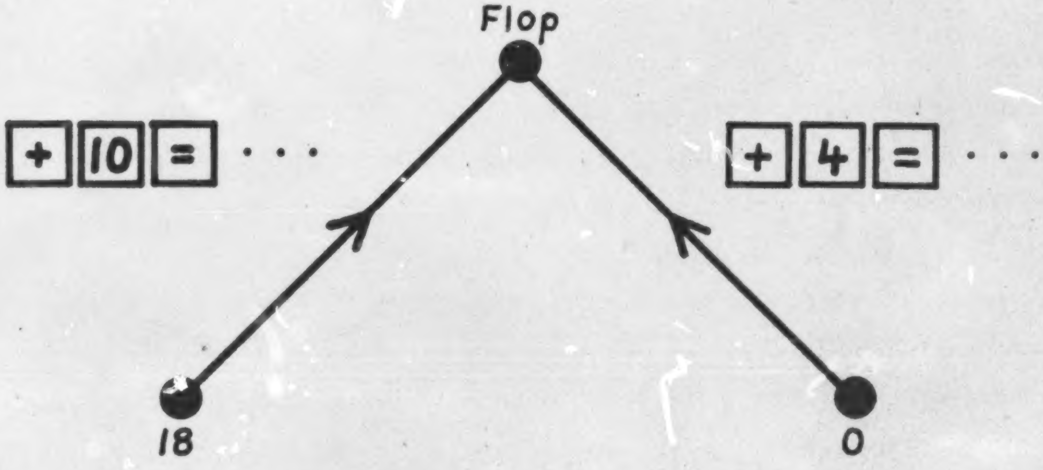
Patience and a well-chosen question to stimulate their thinking are again employed when the students only partially recognize a pattern that has been generated. Once again, the students show they are capable of improving their understanding.

The teacher avoids asking Harry to explain why, "We are counting by 30. ", wishing not to interrupt the train of thought with something which is premature. If students are to be encouraged to trust their intuition about numbers and to develop it, they must not be pushed too quickly into finding the logical reasoning behind the intuition.

The teacher proceeds in much the same way with the second clue. First she presents it and silently lets the students have enough time to absorb it and to sort out their thinking. She then gives them a period of time to play with the clue and to discover which numbers will work and which will not. Instead of using explicit reasoning to prove that 54 and all the larger numbers are not possible, she allows her questions and the boys' trial and error activities to lead them to this conclusion implicitly, which is more appropriate at this stage of their development. Once again, less teaching promotes better learning.

Problem 2: Who is Flop?

Several weeks later, Jack, Max, and Dave are together in a group. The teacher draws this picture on the board and again remains silent.



The students begin working with their hand-calculators. After a few minutes, the teacher asks:

T: Which numbers could Flop be?

Dave: 188 and 288.

T: Which is the smallest number that Flop could be?

Max: 28.

T: Some other numbers?

Jack: 38.

T: Do you agree?

Dave: No, 38 is not a multiple of 4. Flop could be 48.

Max: Then 68 and 88.

Dave: There is a pattern; the numbers end in 8.

Max: The next number goes 2; 4; 6; 8.

T: You mean the next to last digit.

Dave: You add 2 each time.

T: Is it really 2 that you add?

Max: Add 20. You get 88, 108, 128, 148, 168, and so on.

The teacher writes these numbers on the board.

T: Which is the largest number smaller than 1,000 that Flop could be?

Max: 998.

Dave: No, it's 988. The second digit cannot be 9.

T: Which is the smallest number larger than 1,500 that Flop could be?

Max: 1,518.

Jack: No, it's 1,508.

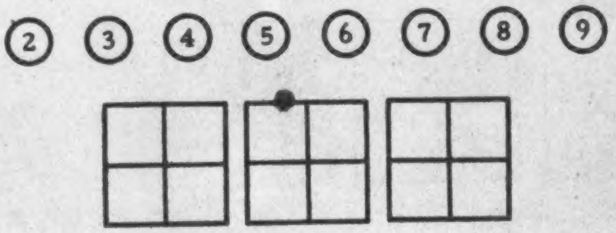
T: The smallest number larger than 850?

Jack: 868.

T: The largest number smaller than 940?

Dave: 928.

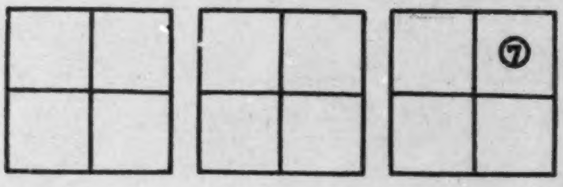
The teacher displays three Minicomputer boards and the weighted checkers:



T: Here is another clue.

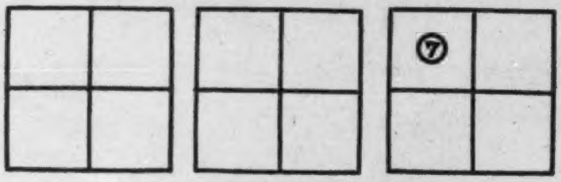
Flop can be put on the Minicomputer using exactly one of these checkers.

Jack puts a ⑦ - checker on the 4-square.



Jack: Flop could be 28.

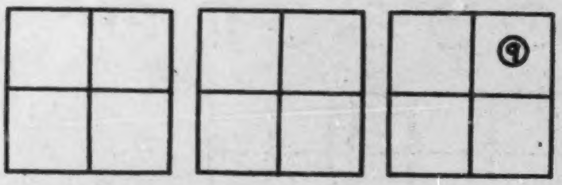
Dave puts a ⑦ - checker on the 8-square.



Dave: Flop could be 48.

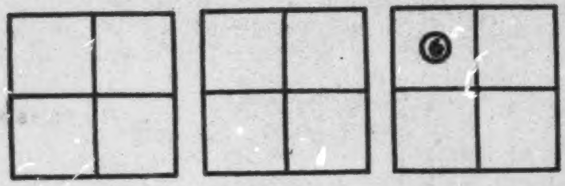
Max: That's not 48; it's 56.

Dave puts a ⑨ - checker on the 4-square.



Max: That's 9 x 4 = 36. So it cannot be Flop.

Jack puts a ⑥-checker on the 8-square.



Jack: 6 x 8 = 48. Flop could be 48.

T: Could Flop be 68? or 88? or 108?

The boys make some attempts to put these numbers on the Minicomputer; they are still not sure about 68 and 88.

T: Which is the largest number you can put on the ones' board using exactly one of these checkers?

Jack puts a ⑨-checker on the 8-square.

Jack: It is 72.

Dave: There's no way of putting on 68 or 88.

T: What if you use the tens' board?

The students make some tries with a checker on the tens' board.

Max: They all end in 0.

Jack: Flop is 28 or 48.

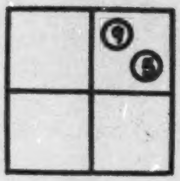
T: I have a third clue.

The teacher points to a ⑤ - checker and to a ⑨ - checker.

T: Flop can be put on the Minicomputer using exactly these two checkers.

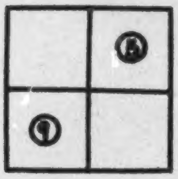
The students try different combinations of the two checkers.

Jack:



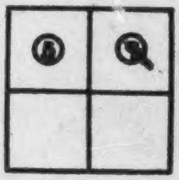
= 56

Max:



= 38

Dave:



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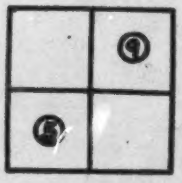
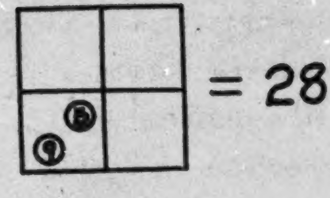
Jack:



= 46

Max:



They unanimously claim that Flop cannot be 48 and conclude that the secret number is 28.

Comments:

As in the session reported first in this section, the teacher lets a well-chosen, simply drawn clue lead the students into thinking about a problem which is slightly more complex than the one they encountered in that previous session. As before, they immediately begin with relatively large numbers, then fill in those which are missing. But this time they discover the pattern almost immediately; furthermore, they do not need the help of a leading question. The teacher only paraphrases their answers ("T: You mean the next to last digit.", etc.) to clarify the response for the others.

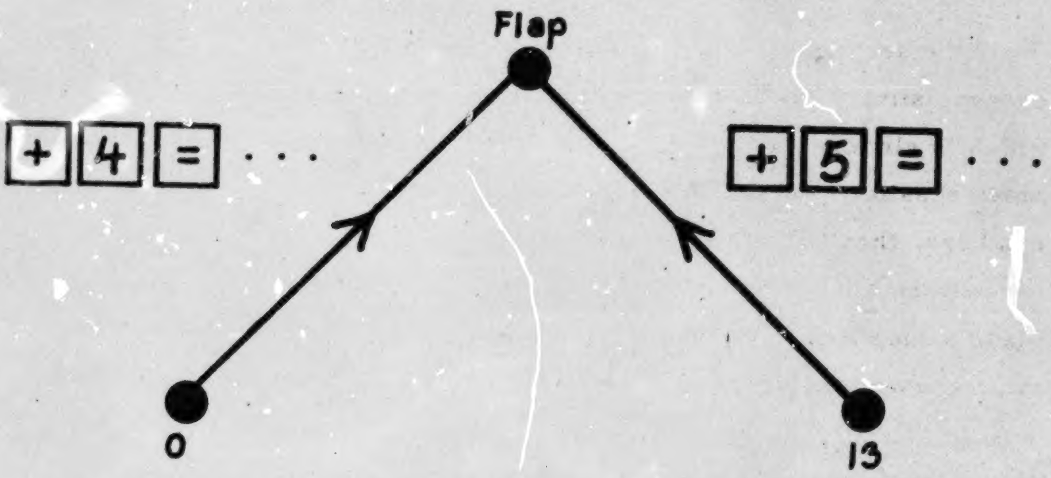
The students are clearly ready to think about problems a little more deeply. With such questions as "Which is the largest number smaller than 1,000 that Flop could be?", the teacher takes them on a playful exploration -- a detour which they obviously enjoy. They give quick decisive answers, and if one person is wrong, another corrects him without help from the teacher.

With the second clue, the students use their "seeing by doing" method of determining which numbers could still be Flop. As they make trials, they explain why the number can or cannot be Flop. Again, they catch their own mistakes and correct each other.

For the third clue, the teacher provides a chance for the students to explore further with the weighted checkers, this time limiting them to two specific checkers. They play with this situation until they are satisfied that 28 is the only possibility for Flop, announcing their discovery with the conviction that is so characteristic of this whole session.

Problem 3: Who is Flap?

Two months later, Max, Harry, and Dave are present at a small group session.* The teacher draws this picture on the board.



The students immediately begin working with their hand-calculators.

Max: Flap could be 88.

Dave: 64.

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*This session was held one week after the session described in Chapter 1 (pp. 42-46) with Harry alone.

T: Do you agree?

Max: No! It doesn't work with the red arrow.

Harry: 1,408.

Max: 28.

Dave: 228 and 2,002,228.

T: Max gave us the smallest number that Flap could be. Which is the next one?

Max: 32, and then 36 and 40.

T: Do you agree?

Dave: No! 53 and then 63.

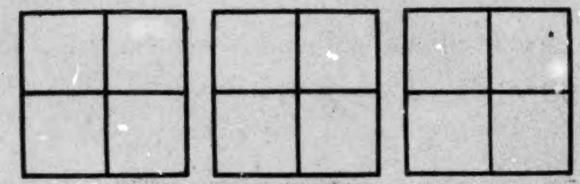
T: Do you agree?

Max: No! I see my mistake. Flap could be 28, 48, 68, 88, 108, 128, . . . going up by 20 each time.

T: That's right. Flap could be any of these numbers, and we can go on adding 20 each time.

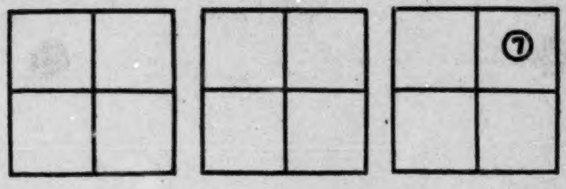
The teacher displays three Minicomputer boards and the weighted checkers:

② ③ ④ ⑤ ⑥ ⑦ ⑧ ⑨



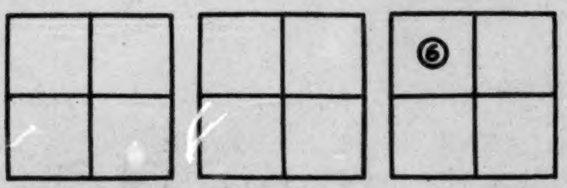
T: My next clue is that Flap can be put on the Minicomputer using exactly one of these checkers.

Dave puts a ⑦ - checker on the 4-square.



Dave: Flap could be 28.

Harry puts a ⑥ - checker on the 8-square.



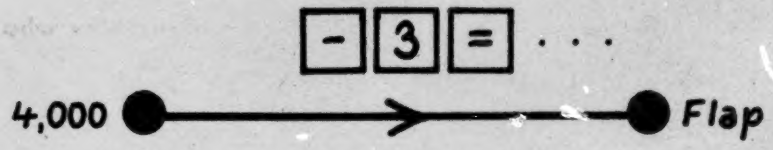
Harry: Flap could be 48.

Dave: You can't get 108 or 128 or 148 and so on, because you can't get 8 on the second board. You just get 0 for the last digit.

T: You mean that if we put the checker on the tens' board, the number will end with 0. What about 68 and 88?

After several tries, the boys are convinced that Flap cannot be 68 or 88. They conclude that Flap is 28 or 48.

For the third clue, the teacher draws this picture on the board.



Max: I think Flap is 48.

But Max cannot explain why.

With undistractable concentration, Harry works on this clue and after a little while, he claims: "Flap is 28."

T: How do you know?

Harry: I started with 4,000 and pressed $\boxed{-}$ $\boxed{54}$ $\boxed{=}$ . . . . I got 4. Then I pressed $\boxed{+}$ $\boxed{3}$ $\boxed{=}$ . . . and I got 28 and 49, but not 48.

T: Why did you choose 54?

Harry: It is a multiple of 3 and it doesn't take so long.

T: Harry has found an interesting way of solving our problem.

Comments:

For the third detective story in this section, the teacher chooses a more sophisticated first clue by substituting a $\boxed{+}$ $\boxed{5}$ $\boxed{=}$ ... arrow for the simpler $\boxed{+}$ $\boxed{10}$ $\boxed{=}$ ... arrow. Although the students show excellent intuition about the pattern when they spontaneously give a series of right answers (ranging from 28 to 2,002,228!), they also display the sudden lapse of understanding that is typical of children with their learning styles. But after a couple of false starts where they consider only one arrow at a time, they are put back on the right track by Max who thinks about both arrows long enough to establish the pattern.

The second clue proceeds much as it did in the earlier sessions, but we observe a clear sign of mental growth. Dave backs up his intuition about the limits of the tens' board with an explanation of why only certain numbers are possible on it. The teacher clarifies his reasoning for the other students by paraphrasing.

The third clue takes them off in another direction; it is the most advanced clue of all. Harry shows the depth of his understanding of the arrow language by substituting a large multiple of 3 to make it easier to find Flap on the hand-calculator. In addition, he is not stymied by finding himself below the two numbers; he simply uses the return arrow to go forward and find which of the two numbers is Flap.

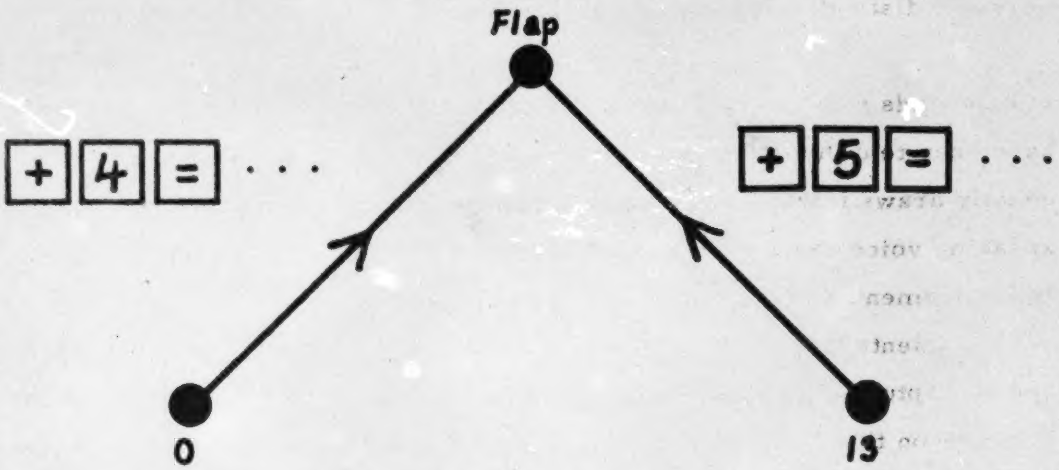
In general, the various detective stories were repeated in all the small groups, giving us several accounts of each story. Because they generally follow a similar pattern it is not necessary to include all these accounts here. For the three detective stories presented in this section, we selected the sessions involving a particular group of boys in order to follow their progress through the series of stories. We would also like to report on another session in which the detective story "Who is Flap?" was presented, because it is an interesting record of two boys who rarely get released time for math therapy sessions. Although they both have good intuition about numbers, their progress in the classwork has been slow. Both boys are distinctly non-conformist, though in different respects.

Luke spends much of the math period daydreaming, so to speak, and we have been told this behavior is typical throughout the school day. He frequently draws fantasy pictures instead of working math problems. His speaking voice has a low, almost inaudible tone, though the pitch is high. His movement, too, is quiet and slow. Generally his involvement with other students is at a level that is immature for his age. When his attention can be captured, he appears to comprehend quickly; but he often gives the impression that he finds it very difficult to communicate what he perceives. Luke participates as little as possible in both classroom discussion and individual desk work. Even in the small group it is difficult for him to focus his attention outside his own thought processes.

On the other hand, Greg is highly verbal, with a loud, often harsh voice. He is almost constantly in motion with movements that are quick and energetic. Even when relatively quiet, he usually taps nervously with a finger or toe, as if nervous energy must continuously be released. Greg is considerably handicapped by severe reading problems and most of his released time is spent with a special school district teacher for help in this area. Consequently, he is usually unable to come to the math play therapy sessions. When he does attend, however, he focuses on the non-verbal activities with an attention that he rarely pays to the pencil and paper approach necessary for workbooks or worksheets. In the collective lessons of the class, he makes an effort now and then, but becomes easily discouraged when he feels the work is out of his range. Sometimes he will contribute nothing for long periods of time.

First Clue

The teacher draws this picture on the board.



Greg: Flap could be 18.

Luke: Flap could be 20.

T: Greg, aren't you forgetting the blue arrow?
Luke, aren't you forgetting the red arrow?

Greg: 28; (and a moment later) 48; 58 . . . (a pause for reflection) no, 68.

T: Why 68 and not 58?

Greg: Because 68 is a multiple of 4 and 58 is not.

T: Can you find some other numbers that Flap could be?

Greg: 78 . . . no, it's not a multiple of 4; 88; 98 . . . no, 108.

Luke: 128.

T: All these numbers end in 8.

Greg: Because 8 is a multiple of 4.

T: Does it have anything to do with the red arrows?

Neither boy can verbalize an answer to this.

T: Some more numbers?

Luke: 148; 168; 188; and so on.

Greg: They are always 20 apart.

T: What are some numbers larger than 500 that Flap could be?

Luke: 508.

Greg: 548.

T: Can you find the largest number smaller than 1,000 that Flap could be?

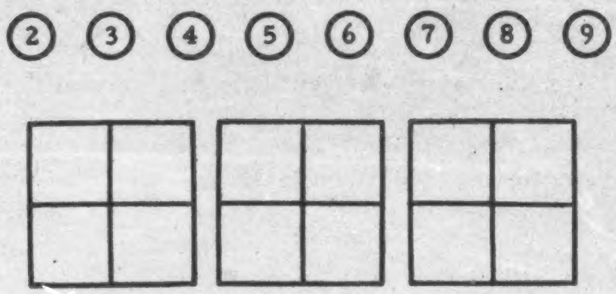
Greg: 998 . . . no, 988.

T: The smallest number larger than 775?

Luke: 788.

Second Clue

The teacher displays three Minicomputer boards and the weighted checkers:

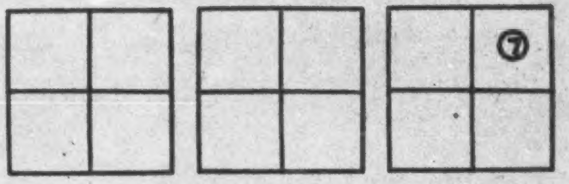


T: Flap can be put on the Minicomputer using exactly one of these checkers.

Greg: 88.

T: Can you put it on the Minicomputer?

Obviously Greg cannot, but he puts a ⑦ - checker on the 4-square.



Greg: Flap could be 28.

After a little while, Greg observes: "You can put 900 on the Minicomputer."

T: That's right. But is 900 a number that Flap could be?

Greg: No.

Luke: Is there a ⑰-checker?

T: No, only the checkers from ② to ⑨.

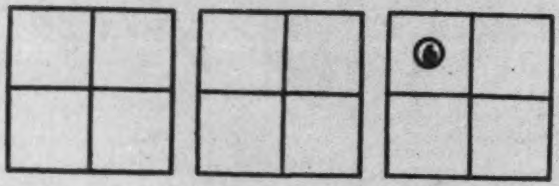
Luke: I could have gotten 68 with a ⑰-checker on the 4-square.

At this point, the two boys work independently and in two different directions.

Luke is trying to find some way of putting 68 on the Minicomputer. Greg is focused on the tens' board. He tries 9 x 10, 9 x 20, 9 x 40, 9 x 80; then he selects another checker and tries 5 x 10, 5 x 20, ... The teacher lets each boy follow his own train of thought for awhile.

T: Luke, you didn't find a way of putting 68 on the Minicomputer. What about 48?

After several tries, Luke puts a ⑥ - checker on the 8-square.



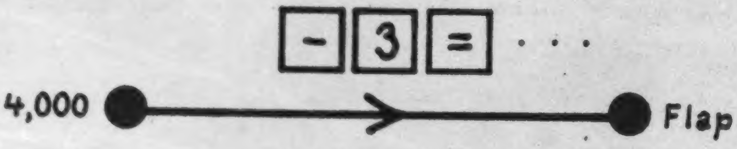
Greg: That's 48. Flap could be 48.

The two boys seem to be tired. So the teacher decides to stop the research on the second clue by simply saying:

T: We have found that Flap could be 28 or 48.

Third Clue

She draws this picture on the board.



After a very short time, Greg claims: "Flap is 28."

T: How do you know?

Greg: I started with 4,000, pressed $\boxed{-}$ $\boxed{3}$ $\boxed{=}$ . . . , and got 3,928. I know I'll also get 28.

Greg cannot verbalize a complete explanation of his discovery, but he has a strong conviction.

Comments:

Although the material covered was the same, this session differs from the preceeding one in several ways. Initially, neither Greg nor Luke is able to find a single possibility for Flap, because they concentrate only on the red arrow or on the blue arrow. Once Greg gets pointed in the right direction, however, he identifies the pattern, correcting himself when in error and getting more frequent assistance from Luke as he goes along. In spite of the fact that they both show an instinctive grasp of the situation, Greg can only partially verbalize the rationale behind it and Luke cannot add anything helpful at this point. Yet, when given a chance to play with the clue as the other group of three boys did, they too find several good answers (508; 548; etc.).

The second clue activates an extremely interesting situation. After a mixture of wrong and right answers, each boy takes a different approach and becomes intently concentrated on his own course, so much so that he cannot be deflected. One has the feeling of holding the reins of a team of horses who have suddenly turned onto separate paths.

The teacher "gives them their head" for a short time, then directs them both back to the problem at hand. She observes that the detours have tired them and does not insist that they exhaust the possibilities for this clue, but rather moves on to the next. It is important to be aware when students have reached a saturation point with one clue in order to avoid losing them before the story is finished because they have become overly tired.

Greg shows excellent intuition about the third clue, but cannot verbalize why he is sure Flap is 28. Once again, the teacher does not insist upon a reasonable explanation, but accepts the intuitive answer, understanding that he has made the important first step and allowing him to feel satisfied with that.

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[A] Axline, Virginia M., Play Therapy. New York: Ballantine, 1969.

[M] MacCracken, Mary, A Circle of Children. New York: Signet, 1975.